

CHAPTER 1

Möbius inversion and the Principle of Inclusion-Exclusion (PIE) are discrete derivatives

Möbius inversion and the Principle of Inclusion-Exclusion (PIE) are discrete derivatives on particular posets, analogs of (backward) discrete difference. It's just order theory, and it algebraicizes as "incidence algebras", analogs of group algebras.

The picture is:

- discrete difference = line

Poset:

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \dots$$

Difference kernel

(what you pair a function against to differentiate):

$$0 \quad -1 \quad 1$$

- PIE = hypercube

Poset

$$\{a\} \longrightarrow \{a, b\}$$

$$\uparrow \qquad \qquad \uparrow$$

$$\{\} \longrightarrow \{b\}$$

Difference kernel

$$-1 \quad 1$$

$$1 \quad -1$$

- Möbius inversion = hyperbox

(several steps in each direction), and is a combo of the above

Poset

$$3 \longrightarrow 6 \longrightarrow 12$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$1 \longrightarrow 2 \longrightarrow 4$$

Difference kernel

$$0 \quad -1 \quad 1$$

$$0 \quad 1 \quad -1$$

Möbius inversion seems mysterious at first because it is presented as a (Dirichlet!) convolution operator (likely the first convolution a novice reader has seen), and "square-free" sounds like it has subtle number-theoretic meaning (it doesn't: it's just the order picture above).

We review each case, emphasizing the connections.

1. Discrete difference

Given a sequence a_0, a_1, \dots , one can integrate it, getting the sequence of partial sums:

$$(\Sigma a)_n := \sum_{i \leq n} a_i$$

Inverse to this (corresponding to differentiation) is successive differences:

$$(\Delta b)_n := b_n - b_{n-1}$$

(and $(\Delta b)_0 = b_0$).

These two operations are inverse, which is the discrete fundamental theorem of calculus: $\Delta \Sigma$ is the difference of partial sums, while $\Sigma \Delta$ is the telescoping sum.

2. PIE

fixme: The classic PIE is subtle from this POV, and I don't really understand it; see `TEX` source for details.

Given a function on a power set $f: \mathcal{P}(X) \rightarrow \mathbb{R}$, one can integrate it:

$$(\int f)(A) := \sum_{B \subset A} f(B)$$

Inverse to this is the XXXXXXXXXXXXXXXX

We illustrate with the classic application of the PIE as counting derangements (fixed-point-free permutations). The idea is that “permutations with fixed set A ” is the derivative of “permutations that stabilize A (and possibly more besides)”, and the latter is easy to count (it's $(n - k)!$, where $k = |A|$); note that the derivative is in the *opposite* of the usual order: $A \leq B$ iff $A \supset B$.

Formally, a permutation σ stabilizes A iff the fixed set of σ *contains* A : $A \subset \text{Fix}_\sigma$. The permutations that fix A are the identity on A , and arbitrary on $X \setminus A$, so they are exactly $S_{X \setminus A}$, and there are $|X \setminus A|! = (n - k)!$ of them.

Given $A \subset X$, $\text{Fix } A$ are all disjoint $\text{Stab } A = \coprod_{B \supset A} \text{Fix } B$

to count derangements which fix particular subsets: let F_A be the set of permutations that fix $A \subset X$, look at $\mathcal{P}(X)$, and OH! The point is that you have two functions: everything fixes \emptyset , but we want to count those that fix *exactly* the empty set

Incidentally, how many permutations have fixed set A for $A \neq \emptyset$? That's just a derangement of $X \setminus A$, so it's $[(n - k)!/e]$.

3. Incidence algebras

The notion of an incidence algebra algebraicizes this: define a product on the module of “functions on (intervals in) a poset” via convolution, and then the differentiation (and integration) operators become convolution operators.

Formally:

¹

Most familiar interesting application of PIE is counting derangements (fixed-point-free permutations).

BTW, why is Möbius inversion interesting? Combinatorially it’s cute but not deep. It’s interesting b/c summation (of arithmetic functions) is interesting (related to totient, divisors, zeta function)

Wiki:

See wiki page on incidence algebras ...and wikify this

convolution: The formula $(f * g)(x) = \int_{yz=x} f(y)g(z)$...makes sense without any *group* structure, and you don’t even need a complete operation. (partial magma)

wiki: - link semigroup algebra / monoid algebra to group ring, and discuss there

! you can’t in general differentiate ! EG, in a group algebra, the group of units is $\{\pm g\}$ (notably, ζ is not invertible) Concretely, in $\mathbf{Z}[\mathbf{Z}/2]$, $\zeta(1 - g) = 0$.

¹Why $a \leq b$? You might consider $a < b$ (strictly less), but this isn’t as good, as it’s not reversible. (It does work for functions on the naturals though, where you can use forward differences instead of backward differences, but this is unusual.) For instance, you never capture the value of top elements, and bottom elements are indistinguishable from the empty set: if you “integrate” (via $a < b$) a function on the poset with a single element, you get zero.