# INTRO TO DUALS OF VECTOR SPACES 

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To Glenna


#### Abstract

This note explains dual vector spaces in some depth, to a bright student without much hard-core abstract math background; say, a chemist.


## 1. Briefly

$\mathrm{V}^{*}$, the dual to a vector space, is defined by $\mathrm{V}^{*}:=\operatorname{Hom}(\mathrm{V}, \mathrm{K})$, i.e., the (contravarient) functor represented by the base field. We'll explain in more detail, and give far more intuition.
1.1. Asking the right questions. The answer to the question "what is the dual space" is brief; it's another question to put it in context and understand why we care, how to work with it. Context and generality are essential because if you only have one example, it's impossible to know which features are specific to that example. For instance, the real numbers are an example of a field - but to understand the real numbers and how they're special, one needs to know what is generally true about fields and what is specific to the real numbers.

It's yet another question to see cool examples and applications, to appreciate the power and usefulness, and to get intuition from working with it.

Hopefully this note is an example of how to ask questions properly, and how to answer them properly. It's quite common, especially in beginning mathematics (and by beginning mathematicians), to be excessively terse: many questions, prima facie, admit a terse answer - but we don't want just answers: we want understanding.
1.2. Example to keep in mind. Duals are particularly confusing either when you try to learn them completely abstractly, or when you're always working in terms of a basis. A good happy medium is to work with the space of polynomials (of a given degree), as we're not too wedded to any particular basis for it, and we can describe concrete elements of its dual. It has enough structure (so that we can work with it), but not too much (so that we get confused) - so long as you forget about the multiplication (grin).

FIXME: You should work through all this using polynomials, and I should write up examples using polynomials everywhere; the key examples are: evaluation at a point, and evaluation of a derivative at a point.

FIXME: also, note that this is a good way of seeing how the dual of a direct sum is a direct product (and the dual of a direct product is much bigger still...): in the basis dual to $1, x, \ldots, x^{n}, \ldots$, "evaluation at $a^{\prime \prime}$ has vector $1, a, \ldots, a^{n}, \ldots$, which doesn't terminate. (and the pairing only makes sense $b / c$ all polynomials terminate; evaluation at a point doesn't make sense for a general formal power series;
indeed, we can define "radius of convergence" in terms of the subspace on which various "evaluate at points" are defined.)

## 2. Coordinates

Vectors in $\mathbf{R}^{n}$ can be thought of as column matrices, while elements of $\left(\mathbf{R}^{\mathfrak{n}}\right)^{*}$, called covectors can be thought of as row matrices.

The main reason for this is that way the evaluation pairing of a covector with a vector is simply matrix multiplication.

More deeply, this reflect the isometry $\operatorname{Hom}(\mathrm{V}, \mathrm{W})=\mathrm{V}^{*} \otimes \mathrm{~W}$ for finite dimensional vector spaces, and illuminates the structure of linear transforms.

You can think of a linear transform as a matrix, which you can break up as a bunch of column vectors, or a bunch of row vectors.

The column vector POV says:
"a linear transform is determined by the vectors in $W$ that it sends a basis of $V$ to"

The row vector POV says:
"a linear transform is determined by the coordinates in $W$ that it evaluates vectors in $V$ to"

Also, note that given a row and a column, if you multiply them the opposite way, you get a matrix! (And not just any matrix - a rank 1 one.) This story goes on for a while; we won't pursue it here.

## 3. SOME CATEGORY THEORY

Why do we do this?
Firstly, the dual is a (contravarient) functor: if you want to understand it, you need to understand functors.

Second, the fact that the double-dual of a finite-dimensional vector space is "naturally" isomorphic to the original vector space is a statement about natural transforms of functors, so you'll need to learn that, too.

Thirdly, linear algebra without category theory is confusing: all the category theory is there, lurking, and if you try to ignore it, you'll suffer. We may as well make it explicit.

Fourthly, category theory is a beautiful and fundamental part of mathematics, showing common algebraic underpinnings of widely disparate areas, and providing important insight and unification.

Lastly, linear algebra is a showcase of category theory - many of the important concepts are here, and it's an easy enough context that you can do a lot of math. I like to say that with easy objects, we can build up complex structures: what's interesting are the connections and interrelations of the objects: with complicated objects, even understanding the objects is difficult and interesting.
lotta category theory use EGs from set theory
(egs of "is this a category?")
functors; e.g.s from set theory (esp. power set - both co-varient and contravarient)
when Hom-sets have structure
the only structure one assumes by default on Hom-sets is a set. sometimes there's more (and what we requires is that composition is a good map) (remember, we can put whatever structure we want on the objects, but category theory only sees the maps)

It's particularly interesting when Hom-sets have the same structure as the category itself (okay, it's -very- interesting: you can use the subject to study itself more) (egs: sets, top, algGeo, and of course linear algebra)
in our case, $\operatorname{Hom}(\mathrm{V}, \mathrm{W})$ has the structure of a vector space, and composition is bilinear! (NB: not linear)
representable functors:
the identity functor is represented in Sets and VectSpace
exercises: show that power set contravarient functor is represented by $\{0,1\}$; (call it subobject classifier; get topoi) give a counting argument with finite sets to show that the power set covarient functor cannot be represented.
(sidebar: words and pictures: commutative diagrams are as rigorous as words; people use words $\mathrm{b} / \mathrm{c}$ they're more familiar, more linear, and easier to typeset. But (mathematical) reality isn't words it just -is-, and words, diagrams - these are just attempts to capture it.

Hom as a bifunctor (cute diagrams - spend a bit of time)
"Naturality" discuss natural transforms of functors (note that clearly dual -cannot- be natural, as it's contra-varient; more concretely, show a diagram that doesn't commute) eg of id -i power set for sets ( $x \mapsto$ $\{x\}$ ) (have trivial EGs too)
the map $\mathrm{V} \rightarrow \mathrm{V}^{* *}$ always exists, and is a natural trans. of functors. (if you deal with topological vector spaces, life gets more interesting) for algebraic vector spaces, it's always injective; hence for finitedimensional vector spaces, always isomorphism, so it's a "natural equivalence of functors"
$\mathrm{V}=\operatorname{Hom}(\mathrm{K}, \mathrm{V}) \rightarrow \operatorname{Hom}\left(\mathrm{V}^{*}, \mathrm{~K}^{*}\right) \rightarrow \operatorname{Hom}\left(\mathrm{V}^{*}, \mathrm{~K}\right)=\mathrm{V}^{* *}$ this is good $\mathrm{b} / \mathrm{c}$ it's purely categorical

Duality of subsets
(this is a good example of a Galois connection)
(I think $S^{*}=\left\{x \in V^{*} \mid x(s) \leq 1 \forall s \in S\right\}$ works) (note that $S^{* *}$ needn't equal $S$, but it always contains it, and $S^{* * *}=S^{*}$ always; this closuretype situation always occurs in Galois connections) (this is useful in Economics, I think - and not well-known in math) (this makes rigorous the "duality of polyhedra", as in cube/octahedron etc. this latter makes a good example (try 2d first) - but note that that's actually a different story! That is, "put a vertex in the center of every face" agrees with this duality, up to rescaling, for regular and semiregular polytopes, but -not- for general ones!)
(proof of properties: dual is inclusion-reversing (pure logic); dual is convex (calculation); $S \subset S^{* *}$ (pure logic); if $S$ is convex, then $S^{* *}=S$ (trickiest bit - several ways to show: concretely, if $x \notin S$ and $S$ is convex, then it's separated by -some- hyperplane, etc. more abstractly, convex means intersect of hyperplanes etc.) (okay, better: it's purely formal: double dual of half-space is the same half space, so they're closed. A convex set is an intersection of halfspaces, and intersection of closed is closed.)

Free things
The similarity between sets and vector spaces is not accidental; vector spaces are "free objects" in some sense (get the details right) in particular, introduce adjoints, esp the Free/Forget adjunction

What -are- duals? (put in the epilogue)
involutions - indeed, even an equality can be considered a duality of a sort - it's two different ways of looking at something. (mention pontryagin duality)

Here's a geometric way of seeing it:
vectors are points in space; equivalently, dim 1 subspaces (lines) and magnitudes (unique except for 0 ): a point on the subspace
covectors are affine hyperplanes; equivalently, codim 1 subspaces and magnitudes You can visualize that as "how far from the origin" the hyperplane is.

To evaluate a covector and a vector, viewed this way: what multiple of the vector lands on the hyperplane? Note that this is bilinear, and well-defined except for when the line and the hyperplane are parallel, meaning the vector is in the kernel of the covector (in particular, this works for 0)

Note that given an inner product of $V$, you get a duality between vectors and covectors (viewed in this way) via perp: perp to a line is a hyperplane.

This is ultimately coming from affine duality (galois connection): $\mathrm{V} \times \mathrm{V}^{*} \rightarrow \mathrm{~K}$, where $\mathrm{x} \mapsto\{v \mid v(\mathrm{x})=1\}$

More ultimately, this comes from projective duality; the reason we get a problem with 0 is $\mathrm{b} / \mathrm{c}$ we're running into the hyperplane at infinity, in some sense (this is a good example of how we get "limits" and "degeneration" in algebraic geometry: visually, take two lines in the plane that are intersecting. If you rotate one, then their intersection will be one point for almost all points, but then will blow up to being the whole line when they coincide.)

The pairing $\mathrm{V} \times \mathrm{V}^{*} \rightarrow \mathrm{~K}$ gives a -host- of dualities, (especially for $K=\mathbf{R}$, as then you get an ordering)!

$$
x \mapsto\{v \mid P\}
$$

...where $P$ is: $v x=0$ : vector duality: closure is vector space closure, dual is perp $v x=1$ : affine duality: closure is affine closure, dual is ...it is what it is: it's "affine 'perp' ". $v x \leq 1$ : convex (vector) duality: closure is convex (vector) closure, meaning you have to include 0 , dual is convex dual.

You can get convex affine duality, but then you have to pair $V$ with $\operatorname{Aff}\left(\mathrm{V}^{*}\right)$. This makes sense: the -affine- dual of a subset V can't be any -particular- subset of $\mathrm{V}^{*}$ : you could move it by any translation, which corresponds to living in Aff instead of the original space. (er, and by Aff I mean: it's what you get if you include translations of $V$, together with linear duals)

Note that affine duality is ultimately coming from projective duality: points in $\mathbf{P V}$ correspond to hyperplanes in $\mathbf{P V}^{*}$.
(and $n$ generic points determine a hyperplane, while $n$ generic hyperplanes determine a point: nicest for $n=2$, where "generic" means "not equal", hence the classical projective duality of the projective plane)

## 4. BASES AND COORDINATES

This section is somewhat long and technical: I explain essentially everything about bases and coordinates in linear algebra.

Note that a very good concrete space to work with when you want a concrete space but not a specified basis is the space of polynomials. In particular, there's lots of ways of describing elements of the dual without using coordinates. It's a very good alternative to $\mathrm{K}^{n}$, which has a confusing basis, and V , which is just abstract.

It's often useful for intuition and concreteness to work things out in terms of a basis. However, you should also be cautious about bases as they obscure the theory and the actual structure: if you can do something in terms of a basis, doing it without choosing a basis will generally clarify matters and deepen your understanding.

The most important theoretical task when doing a construction in terms of a basis is to ask what happens when we change the basis. Why does this matter so much? Because if we do something in terms of a basis, and know what happens when we change the basis, then we know the whole story independent of choice of basis.
4.1. Notation. We potentially have a great many bases at once: to understand all the possibilities, we'll want to understand two spaces, V and W , and maybe two bases on each, and for any basis we also have a dual basis on $\mathrm{V}^{*}$, so we could get 8 bases! It's easy to get confused.

Further, the "correct" mathematical notation can obscure matters further: one often denote the basis $\left\langle e_{1}, \ldots, e_{n}\right\rangle$ by $\mathcal{B}_{e}$ or some such, which is pretty verbose.

There's lots of conventions, especially for duals.
One might use $e_{1}, \ldots, e_{n}$ as a basis for $V$, and $f_{1}, \ldots, f_{m}$ as a basis for $W$.

Or $e_{1}, \ldots, e_{n}$ as a basis for $V$, and $f_{1}, \ldots, f_{n}$ as another basis for $V$.
Or $e_{1}, \ldots, e_{n}$ as a basis for $V$, and $e_{1}^{\prime}, \ldots, e_{n}^{\prime}$ as another basis for $V$.
For duals, one might use $e_{i}$ as a basis for $V$, and $e_{i}^{\prime}$ (or $e_{i}^{*}$, or $f_{i}$, or $\mathrm{f}^{\mathrm{i}}$ ) as the dual basis for $\mathrm{V}^{*}$

FIXME: be consistent below!
Lastly, I use $\phi$ and $\psi$ for the coordinates maps in analogy with manifolds, where we use these for coordinate patches.
4.2. Categorical background. The key to understanding the relationship between (abstract) linear algebra and (coordinate) matrix algebra is:
" $\mathbf{R}^{n}$ and matrices are a skeletal subcategory of vector spaces and linear maps"

Given a category C , a skeletal subcategory S is specified by a collection of objects $\left\{\mathrm{A}_{\mathrm{i}}\right\}$ (called the skeleton), with exactly one from each isomorphism class of $C$. IE, for all object $X \in C$, there is a unique $A_{i}$ such that $X \cong A_{i}$. We let $S$ be the full subcategory on $\left\{A_{i}\right\}$ : that is, the objects are the $\left\{\mathcal{A}_{i}\right\}$, and the maps are all maps ${ }^{1}$ between them.
[Formally: the inclusion of a skeletal subcategory has as adjoint: "send an object to the element of the skeleton that it's isomorphic to". This doesn't specify an adjunction: it's not just two functors, but -also- requires a consistent -identification- of the two Hom-sets. We can provide this by specifying isomorphisms $\phi_{X}: A_{i} \xrightarrow{\sim} X$ for all objects; this is the unit of the adjunction, and this is a good example of how two functors and a unit determine an adjunction.

That is, a skeletal subcategory is reflective (and coreflective), and the isomorphisms are the (co)reflection.]

Given a skeletal subcategory, you can "reduce most questions" to questions inside the skeleton. This statement is intentionally vague: it's rather like "first-order logic": any statement about a specific diagram can be so reduced, but questions about the category as a whole can't be reduced: see below under "Limitations of skeletons".

How does this work?
Given an object $X \in C$, we have $\phi_{x}: A_{i} \rightarrow X$, so any question about the object $X$ can be answered by answering the (transfered) question about its isomorphic object $A_{i}$. EG, the space of (orthonormal) 1frames in $\mathbf{R}^{n}$ is (naturally) homeomorphic to $S^{n-1}$, so the space of 1frames in any real vector space $V$ of dimension $n$ is homeomorphic to $\mathrm{S}^{\mathrm{n}-1}$.

Now given two objects $X, Y \in C$, with $\phi_{X}: A_{i} \rightarrow X$ and $\phi_{Y}: A_{j} \rightarrow$ $Y$, we have

$$
\left(\phi_{X}^{-1} \times \phi_{Y}\right)_{*}: \operatorname{Hom}\left(A_{i}, A_{j}\right) \xrightarrow{\sim} \operatorname{Hom}(X, Y)
$$

so any question about the Hom-object can be reduced to within the skeleton.

For instance, every linear map between finite-dimensional vector spaces has a rank, by defining the rank of a matrix and representing the linear map as a matrix. We can also prove this directly though,

[^0]and this is a good example of how intrinsic definitions offer more insight.

Similarly any other diagram can be reduced into the skeleton.
A caution: note that we also have End $X \cong$ End $A_{i}$ by

$$
\left(\phi_{X}^{-1} \times \phi_{X}\right)_{*}: \operatorname{Hom}\left(A_{i}, A_{i}\right) \xrightarrow[\rightarrow]{\sim} \operatorname{Hom}(X, X)
$$

but that we need to use the same identification each time.
Lastly, note that isomorphisms between two objects can be identified with automorphism of a fixed object:

$$
\operatorname{Iso}(X, Y) \cong \operatorname{Aut}\left(A_{i}\right)
$$

if $A_{i} \cong X \cong Y$. Note that $\operatorname{Iso}(X, Y)$ is a torsor (for $\operatorname{Iso}(Y)$ and for $\left.\operatorname{Iso}(X)^{\text {op }}\right)$, and $\operatorname{Aut}\left(A_{i}\right)$ is a group (recall that a group is exactly a pointed torsor): the point comes from the isomorphism $\phi_{Y} \phi_{X}^{-1}: X \rightarrow$ $A_{i} \rightarrow Y$.
4.3. Matrices as a skeletal subcategory for linear maps. We now work out what this abstraction means for the specific case of linear algebra.

So for vector spaces and linear maps, every vector space V is isomorphic to $\mathbf{R}^{\mathrm{n}}$ for some n : in fact, for $\mathrm{n}=\operatorname{dim} \mathrm{V}$.

Similarly, $\operatorname{Hom}(V, W)$ is isomorphic to $\operatorname{Mat}(m, n)(m \times n$ matrices, IE, $m$ rows and $n$ columns), where $n=\operatorname{dim} V, m=\operatorname{dim} W$ (note the reversal: rows is output, columns is input), via $\mathbf{R}^{n} \cong V$ and $\mathbf{R}^{m} \cong W$, and maps $\mathbf{R}^{n} \rightarrow \mathbf{R}^{m}$ have natural matrix representations.

A choice of isomorphism $\mathbf{R}^{n} \xrightarrow{\sim} V$ is exactly the same as a choice of basis. The correspondence is given by:

- given a basis $e_{1}, \ldots, e_{n}$, we get a map $\mathbf{R}^{n} \rightarrow V$ by $\left(a_{1}, \ldots, a_{n}\right) \in$ $\mathbf{R}^{n} \mapsto a_{1} e_{1}+\cdots+a_{n} e_{n} \in V$. This is an isomorphism by definition of basis.
- given an isomorphism $\phi: \mathbf{R}^{n} \xrightarrow{\sim} \mathrm{~V}$, we get a basis $e_{1}, \ldots, e_{n}$ of $V$ by $e_{1}=\phi(1,0, \ldots, 0), e_{2}=\phi(0,1,0$, dots, 0$)$, etc. - just push the standard basis forward. This is a basis by definition of isomorphism.

Okay, so given a map $T: V \rightarrow W$, how do we express it in terms of a basis, from this point of view? A basis $e_{1}, \ldots, e_{n}$ of $V$ corresponds to $\phi_{e}: \mathbf{R}^{n} \xrightarrow{\sim} V$, and a basis $f_{1}, \ldots, f_{m}$ of $W$ corresponds to $\psi_{f}: \mathbf{R}^{m} \xrightarrow{\sim}$ W.

So we get a diagram

and to get a matrix we need a map $M: \mathbf{R}^{n} \rightarrow \mathbf{R}^{m}$, IE, we need a map

"such that the diagram commutes", IE it doesn't matter which path you follow: the result is the same - this requirement just means that $M$ represents T: it's not some arbitrary map.

This is easy to get: it's

$$
M=\psi_{f}^{-1} T \phi_{e}
$$

just reverse $\psi_{\mathrm{f}}$ ! This formula tells us a lot: see how changing the basis of the domain acts on the right, while changing the basis of the codomain acts on the left - but with an inverse. Each of these is reflected in the nuances of the computations.

Also, note that it says "our input is in terms of the basis $e$; our output is in terms of the basis $\mathrm{f}^{\prime \prime}$ : input comes in on the right and comes out on the left. Yes, this is confusing: it's because we want to write $T(x)$ (we want to write $T$ on the left), which means the $x$ comes in the right side of T and comes out the left: think of $\mathrm{U}(\mathrm{T}(\mathrm{x})$ ). In some places they do composition and matrix multiplication the other way: $(x) \mathrm{T}$ and $((x) \mathrm{T}) \mathrm{U}$. (This especially makes sense in languages that put the verb last, like Deutsch, I think; our $f(x)$ comes from putting the verb before the object.)

Now write $T$ in terms of other bases, $e^{\prime}$ and $f^{\prime}$, getting $M^{\prime}$; clearly $M^{\prime}=\psi_{f^{\prime}}^{-1} T \phi_{e^{\prime}}$ How can we get $M^{\prime}$ from $M$ ? Let's look at fixing the base on V : algebraically, we need to multiply M on the right by $B_{e^{\prime} \rightarrow e}=\phi_{e}^{-1} \phi_{e^{\prime}}$.

How can we interpret this? $\mathrm{B}_{e^{\prime} \rightarrow e}$ is the identity on V , with input written in basis $e^{\prime}$, and output written $e$.

Note that this is just a map $\mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ : we cannot interpret it as a map $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{V}$, written in some basis $\left\langle\mathrm{g}_{\mathrm{i}}\right\rangle$, because it would have to be of the form: $\phi_{g}^{-1} \mathrm{~T} \phi_{\mathrm{g}}$.

However, what makes this confusing is that this very matrix admits two more interpretations, if we interpret it as a map $\mathrm{V} \rightarrow \mathrm{V}$ in the basis $\left\langle e_{i}\right\rangle$ or $\left\langle e_{i}^{\prime}\right\rangle$ :

In the basis $\left\langle e_{i}\right\rangle$, we get:

$$
\mathrm{B}_{e^{\prime} \rightarrow e}=\phi_{e}^{-1} \phi_{e^{\prime}}=\phi_{e}^{-1} \phi_{e^{\prime}} \phi_{e}^{-1} \phi_{e}
$$

In the basis $\left\langle e_{\mathfrak{i}}^{\prime}\right\rangle$, we get:

$$
\mathrm{B}_{e^{\prime} \rightarrow e}=\phi_{e}^{-1} \phi_{e^{\prime}}=\phi_{e^{\prime}}^{-1} \phi_{e^{\prime}} \phi_{e}^{-1} \phi_{e^{\prime}}
$$

IE, the map $T_{e \rightarrow e^{\prime}}: V \rightarrow \mathrm{~V}$, where $\mathrm{T}=\phi_{e^{\prime}} \phi_{e}^{-1}$, such that $\mathrm{T}\left(e_{\mathrm{i}}\right)=e_{\mathrm{i}}^{\prime}$ (it sends $e_{i}$ to the $i$ th basis vector of $\mathbf{R}^{n}$, which it sends to $e_{i}^{\prime}$ )

FIXME: ack!!!!! Okay, so this matrix tells you how to rewrite coordinates for $e^{\prime}$ as coordinates for $e$, but it represents (in both the basis $e$ and the basis $e^{\prime}$ ???) the map that sends $e \rightarrow e^{\prime}$ ???

We work this out in much, much more detail below.
4.4. Dual of a basis. One interpretation of duals are as "coordinates".

More correctly: "a element of $\mathrm{V}^{*}$ is a coordinate on V , and a basis for $\mathrm{V}^{*}$ is a system of coordinates on $\mathrm{V}^{\prime \prime}$

Given a basis $e_{1}, \ldots, e_{n}$ of $V$, we get a dual basis $f_{1}, \ldots, f_{n}$ of $V^{*}$, given by:
" $f_{i}(v)$ is the coefficient of $e_{i}$ when you express $v$ in terms of the basis $e_{1}, \ldots, e_{n}{ }^{\prime \prime}$

In symbols, if $v=a_{1} e_{1}+\cdots+a_{n} e_{n}$ (where $a_{i} \in K$ ), then $f_{i}(v)=a_{i}$.
Note that each $f_{i}$ depends on the entire basis $e_{1}, \ldots, e_{n}$ :
"you can't take the dual of a vector - only the dual of a basis"
More formally, if you change a basis, leaving one vector fixed, the dual change of the dual basis needn't leave the dual vector fixed.

This will be clearer when we discuss the effect of change of basis on the dual.

This point is particularly important in differential geometry: $\mathrm{d} x_{i}$ is canonical, but $\partial / \partial x_{i}$ isn't (it depends on the whole basis - IE, it's kinda bad notation)
4.5. Duals in coordinates. NB: dual map is transpose
4.6. Change of basis on a vector space. Given two bases $e_{1}, \ldots, e_{n}$ and $f_{1}, \ldots, f_{n}$ for a vector space $V$, we can ask how to switch between the two.

That, if $v=a_{1} e_{1}+\cdots+a_{n} e_{n}=b_{1} f_{1}+\cdots+b_{n} f_{n}$, then how can we express $\left\langle b_{i}\right\rangle$ in terms of $\left\langle a_{i}\right\rangle$ ? This is a question about the dual space, IE, it's about coordinates, so we will demur. You'll thank me for this - this can get pretty confusing, with all the back and forth.

FIXME: Let's first write the matrix $M_{e \rightarrow f ; e ;}$ that is, the matrix that takes $\left\langle e_{\mathfrak{i}}\right\rangle$ to $\left\langle f_{\mathfrak{i}}\right\rangle$, expressed in the basis $\left\langle e_{i}\right\rangle$. That is, $\mathrm{T}_{e \rightarrow f}$, the linear transform of $V$ defined by $T\left(e_{i}\right)=f_{i}$ is well-defined; we're just writing it in a basis.
4.7. Classification of maps $V \rightarrow W$. We will illustrate change of basis by recalling a classification theorem, and showing the analog for duals.

Theorem 4.1. Recall that maps $\mathrm{V} \rightarrow \mathrm{W}$ are "classified" by their rank. In coordinates, this says that if $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ has rank k , then there are bases for V and W such that the matrix for T has the form like:

$$
\left(\begin{array}{lllll}
1 & 0 & & & \\
0 & 1 & 0 & & \\
& 0 & 1 & 0 & \\
& & 0 & 0 & 0
\end{array}\right)
$$

$I E$, it has k ' 1 's on the diagonal, followed by ' 0 's on the diagonal, and ' 0 's everywhere else.

Proof. The proof is by row and column reduction. That is, if $\operatorname{dim} \mathrm{V}=$ $n$ and $\operatorname{dim} W=m$, then a choice of basis for $T$ expresses it as an $m \times n$ matrix: $m$ rows (because that's the output), and $n$ columns (because that's the input).
mneumonic for left and right: left action is by row operations; right action is by column operations left action act on column vectors; given a set of vectors, it acts on each column in the same way in particular, it doesn't change the columns around, hence it must be acting by row operations dually for a right action
intrinsically defined
A better formulation of the above result is:
Definition 4.1. Say that two diagrams $\mathrm{T}: \mathrm{V} \rightarrow \mathrm{W}$ and $\mathrm{T}^{\prime}: \mathrm{V}^{\prime} \rightarrow \mathrm{W}^{\prime}$ are equivalent if there are isomorphisms $\phi: V \xrightarrow{\sim} V^{\prime}$ and $\psi: W \rightarrow W^{\prime}$ such that this diagram commutes:


Even more formally, consider the category of functors

$$
(\cdot \rightarrow \cdot) \rightarrow(\text { Vect, LinMap })
$$

where objects are functors between these categories and morphisms are natural transforms of functors. This is called the quiver category, because it's a bunch of arrows.

Concretely, an object is just "put a vector space at each dot, and a map on each arrow", and a map is "maps between the corresponding objects such that everything commutes".

Then two diagrams are equivalent if and only if they are isomorphic in the quiver category.
first do for two separate spaces, then for one space (Jordan is harder) (now that I have language of quivers, life is good)

## 5. Vectors, Covectors and Transposes

(cute way of visualizing the duality: horizontal vectors and vertical vectors. Note that this requires a basis cute: this is actually identifying $\mathrm{V}=\operatorname{Hom}(\mathrm{K}, \mathrm{V})$ and $\mathrm{V}^{*}=\operatorname{Hom}(\mathrm{V}, \mathrm{K})$ - that's what the matrices mean) (transposes as making sense once you have a basis) (note that adjoints always make sense, even for square matrices!)
(note that you can always glue on a one-dimensional vector space ;-)
with this, you can deal with functors, but not with such things as $\mathrm{V} \oplus \mathrm{V}^{*}$
also: get bases for symmetric power, alt power, and even other schur functors!
5.1. Quadratic forms as matrices. Lastly, note that quadratic forms can be represented as matrices, but the change of basis rule differs (physicists say: "they transform differently"): you conjugate $M \mapsto$ $A^{\top} M A$, rather than $A^{-1} M A$, as you do for endomorphisms. As a result, the classification is completely different. Rather than Jordan form, over an algebraically closed field the only invariant is rank (er, dim minus max isotropic dim (is this word for "space on which restricts to $0^{\prime \prime}$ ?). Over the reals, you get signature (aka, index: see Sylvester's theorem). Over $\mathbf{Q}$ and $\mathbf{Z}$, life is much more interesting.

Note in particular that our usual invariants of matrices, like trace and determinant and all, are not invariant - though if $A^{\top}=A^{-1}$, then conjugation by $A$ doesn't change 'em. Hence the characteristic poly etc. reflects the metric properties of our basis, or rather of the quadratic form $\mathrm{w} / \mathrm{r} / \mathrm{t}$ that basis.

These are pretty easy to understand: the determinant is the square of the (absolute value of the) volume of the basis, $w / r / t$ the form (IE, the form defines a length, so the parallelepiped defined by the
basis has a volume). Proof is immediate: if it's an inner product, then $M=A^{*} I A$, so $\operatorname{det} M=\operatorname{det} A^{*} \operatorname{det} A=(\operatorname{det} A)^{*} \operatorname{det} A$.

Similarly, the trace is the sum of the (absolute values of the squares of) norms of the basis. For the other symmetric polys, it's similar: sum of the (abs val of squares of the) areas of the squares determined by pairs of basis vectors, etc.

Note that this is kinda goofy if your form is indefinite; also, I'm not sure how to interpret minimal polynomial, or directly interpret the characteristic poly.
5.2. Limitations of skeletons. The following is rather picky, but matters in some contexts, and for reasons of philosophy.

In a nutshell, $\cong \neq=$.
It's not okay to just use $\mathbf{R}^{n}$ and not use abstract vector spaces, but the reason why is a bit subtle, and is categorical. However, it should feel wrong to think of $\mathbf{R}^{n}$ and $\left(\mathbf{R}^{n}\right)^{*}$ as "the same space" - this is a good gauge of how comfortable you are with duals. From the coordinates point of view, the former consists of vectors, while the latter is covectors - so for instance there's a natural pairing of $\mathrm{V}^{*}$ and V , but not of V and V . More sophisticated is that * is a contravarient functor, so there's something weird here.

The difference is ultimately between $=$ and $\cong$ : an isomorphism is not the same thing as the identity.
concrete example: direct sum is "not commutative on the nose" (but is associative) (same issue for tensor product)

In topology this destinction between $=$ and $\cong$ (and "homotopic to") is important: H-spaces, $A_{\infty}$, loop spaces, etc.
quadratic forms, symmetric, etc. maps preserving a basis are lame; basis gives inner product different bases give same inner product ...fibres are an $\mathrm{O}(\mathrm{n})$ foliation of $\mathrm{GL}(\mathrm{n})$...and get $\mathrm{O}(\mathrm{n})$ as subgroup preserving an inner product a choice of basis gives $S_{n}<O(n)$ (heh: a clever student might ask: does every $S_{n}<O(n)$ correspond to a basis? This is a very good question: it leads to representation theory, IE, how can groups (of symmetries) be realized as linear symmetries, IE symmetries of a vector space. Oh, and the answer is no - there's lots of other interesting ways to realize $S_{n}$ as a subgroup of $O(n)$, and the connection between $S_{n}$ and $O(n)$ is very deep indeed; the "best answer" so far is called "Springer theory".) affine spaces: space of frames, etc.

## 6. EXAMPLES AND APPLICATIONS

6.1. Polynomials, the Rational normal curve, and Algebraic Geometry. A great example of spaces with several natural choices of basis is $\mathcal{P}_{\mathrm{n}}$ and $\left(\mathcal{P}_{\mathrm{n}}\right)^{*}$, the space of polynomials of degree at most $\mathfrak{n}$ and its dual (this discussion obviously extends to $\mathcal{P}_{\infty}$ and $\left(\mathcal{P}_{\infty}\right)^{*}$, the space of polynomials of any degree, but then you have to be more careful as it's infinite-dimensional). For the base field, think of $\mathbf{Q}, \mathbf{R}$ or $\mathbf{C}$ : if we have a field of positive characteristic, life becomes rather more complicated ${ }^{2}$.

A natural enough basis for $\mathcal{P}_{n}$ is $\left\{1=x^{0}, x=x^{1}, x^{2}, \ldots, x^{n}\right\}$, which has dual basis "take the kth coefficient", IE

$$
\left\{a_{0} x^{0}+a_{1} x^{1}+a_{2} x^{2}+\cdots+a_{n} x^{n} \mapsto a_{k} \mid 0 \leq k \leq n\right\}
$$

we will refer to this basis as "coefficient coordinates". Note that $\operatorname{dim} \mathcal{P}_{\mathrm{n}}=\mathrm{n}+1$.

Another natural basis (or rather, way to produce bases) for $\mathcal{P}_{n}^{*}$ is: evaluate at $n+1$ points. Remember that a polynomial of degree $n$ is determined by its value at $\mathfrak{n}+1$ distinct points (this generalizes "two points determine a line" to higher degree; note that there's another way of generalizing: "three points determine a quadratic", but also "three points determine a plane": degree or dimension (or both!) can go up), so given $\left\{\alpha_{0}, \ldots, \alpha_{n}\right\}$ (yeah, we use $\alpha_{0}$ to avoid having to write $\alpha_{n+1}$ ) distinct points, we get the basis

$$
\left\{p(x) \mapsto p\left(\alpha_{k}\right)\right\}
$$

for $\mathcal{P}_{n}^{*}$.
Exercise 6.1. Write out the corresponding dual basis for $\mathcal{P}_{n}$.
The choice of $n+1$ points seems a bit weird - after all, can't we evaluate a polynomial at any point in the base field? Yes, of course we can, and that yields a map $K \rightarrow\left(\mathcal{P}_{n}\right)^{*}$, given by sending $\alpha$ to evaluation at $\alpha$.

In coefficient coordinates, $\alpha \mapsto\left(1, \alpha, \alpha^{2}, \ldots, \alpha^{n}\right)$, which is a natural enough map in itself.

This map is called the rational normal curve, and is one of the Most Beautiful Objects in mathematics. It has an absurd number of cool properties, and is a basic object in algebraic geometry. Here's one: any $n+1$ points on it give a basis for $\mathcal{P}_{n}^{*}$ (as a polynomial of degree $\leq$ $n$ is determined by its value on any $n+1$ points), and it's essentially the only such curve!

[^1]Note that it's not unexpected that we should be seeing algebraic geometry: we're talking about polynomials.

However, algebraic geometry also arises naturally from studying vector spaces and linear algebra (the following is intentionally sketchy): essentially, an element of the dual space is a "homogeneous linear polynomial" (IE, polynomial of degree 1 with no constant term) on a vector space, and quadratic forms are (homogeneous) quadratic polynomials on a space.

After all, what is a polynomial? It's a certain kind of function on a space, so it takes in a point on the space and spits out a number. To further characterize polynomials in the set of functions, we should have a notion of "linear" polynomials (constants make sense without any further structure though) - and that's exactly what the dual space does. So what's a "quadratic" polynomial? It's a product of linear ones, or rather a sum of products.

This leads to the study of $\mathrm{Sym}^{n} \mathrm{~V}^{*}$ and so forth, which is another story, but the moral is simple: when you have a linear structure, you have a notion of polynomial: linear algebra leads naturally to algebraic geometry.
7. Epilogue: SO What is the dual space?

The dual space, while deceptively simple, has many useful characterizations.
$\mathrm{V}^{*}$ is....... $\operatorname{Hom}(\mathrm{V}, \mathrm{K})$
...coordinates on $V$
...hyperplanes in V [problem at zero] (dual to vectors in V)
... row vectors (to V's column vectors)
$\ldots$...homogeneous linear polynomials on $\mathrm{V}^{*}$


[^0]:    ${ }^{1}$ In the notion of a subcategory, we also want to allow ourselves to throw out maps. So for instance the category of vector spaces and injective linear maps is a subcategory of vector spaces and all linear maps

[^1]:    ${ }^{2}$ For instance, over $\mathbf{F}_{q}$, the polynomials $x$ and $x^{q}$ evaluate to the same value for every $\alpha \in \mathbf{F}_{\mathbf{q}}$. Weird, eh?

