DIFFERENTIALS, THE MEAN-VALUE THEOREM, AND THE FUNDAMENTAL THEOREM OF CALCULUS

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The Mean-Value Theorem (MVT) and the Fundamental Theorem of Calculus (FTC) are both related refinements of the principle of approximation by differentials. This idea is not new, but it is missing from calculus textbooks. This is a shame, because it provides very easy motivation for these two fundamental results, both of which tend to puzzle students.

Let $f(x) : \mathbf{R} \to \mathbf{R}$ be a differentiable function and $a \in \mathbf{R}$.

(1) **(Definition of derivative)** $$\lim_{b\to a} \frac{f(b)-f(a)}{b-a} = f'(a).$$

This lets us compute f' at a point from values of f at nearby points. The usefulness of calculus arises from turning this around, and using knowledge of $f'(a)$ to reach conclusions about the values of f at these nearby points. The simplest formulation of this is

(2) **(Approximation by differentials)** $$\frac{f(b)-f(a)}{b-a} \approx f'(a), \quad \text{for } b \text{ close to } a.$$

The cost of removing the "lim" is the mutation of "$=$" into "$\approx$" in the equation. Drawing conclusions about f is much easier if we somehow mutate "$\approx$" back into "$=$". This is the content of

(3) **(MVT)** $$\frac{f(b)-f(a)}{b-a} = f'(\xi), \quad \text{for some } \xi \text{ between } a \text{ and } b.$$

This is sufficient precision to let us draw conclusions about whether f is locally increasing or decreasing, or has a local extremum, for instance. But we have really not eliminated the uncertainty in equation (2), merely chased it into the form of the mysterious ξ. We get rid of it altogether by

(4) **(FTC—average form)** $$\frac{f(b)-f(a)}{b-a} = (\text{average of } f'(x) \text{ over } [a,b]).$$

Since the average of a function over an interval is its (definite) integral divided by the length of the interval, this is equivalent to

(5) **(FTC—usual form)** $$f(b)-f(a) = \int_a^b f'(x)\,dx.$$

It is this formula which relates the evaluation of definite integrals to the finding of antiderivatives.

We now ask the question "at what rate does equation (5) change as b is varied?". To answer it, we take the derivative of both sides with respect to b. Since we are used to calling the varying quantity x, we rename b to x and change the dummy variable in the integral to t to avoid confusion. Applying equation (1), we obtain

$$\frac{d}{dx}\Big(\int_a^x f'(t)\,dt\Big) = f'(x)$$

Date: December 26, 1999.

It turns out that this is true if $f'(x)$ is replaced by any continuous function $g(x)$, even when $g(x)$ is not known to be the derivative of some other function.

$$\text{(6)} \qquad \textbf{(FTC—the “other half”)} \qquad \frac{d}{dx}\Big(\int_a^x g(t)\,dt\Big) = g(x).$$

This in particular shows that any such $g(x)$ is in fact the derivative of a function, namely the one defined by integrating $g(x)$ with a varying right endpoint. This constructed antiderivative of $g(x)$ is called its indefinite integral.

It is very easy to prove equations (4) and (5) using equation (3), the MVT. Let $a = x_0 < x_1 < \cdots < x_n = b$ be the regular n-partition of $[a,b]$, so that $x_k - x_{k-1} = (b-a)/n$. Apply equation (3) to f in each of the subintervals $[x_{k-1}, x_k]$ to get

$$\begin{aligned} f(b) - f(a) &= \big(f(b) - f(x_{n-1})\big) + \big(f(x_{n-1}) - f(x_{n-2})\big) + \cdots + \big(f(x_1) - f(a)\big) \\ &= (b - x_{n-1})\,f'(\xi_n) + (x_{n-1} - x_{n-2})\,f'(\xi_{n-1}) + \cdots + (x_1 - a)\,f'(\xi_1) \\ &= \frac{b-a}{n}\,\big(f'(\xi_n) + \cdots + f'(\xi_1)\big). \end{aligned}$$

Here each ξ_k is some point in $[x_{k-1}, x_k]$. By letting $n \to \infty$ we get equation (5) via the Riemann definition of the definite integral, or equation (4) if we move the n in the denominator under the sum of the $f'(x_k)$'s.

In calculus textbooks, the conventional order of presenting the ideas surrounding the FTC is as follows: (a) the definite integral, (b) the indefinite integral, (c) the FTC, first equation (6) and then (5), (d) the average of a function as an application of the integral. The above approach suggests a different order: (a) the definite integral, (b) the integral as an average, (c) the FTC in the order given, (d) the indefinite integral.

In the conventional order, equation (5) is derived from equation (6) by using MVT to conclude that $\int_a^x f'(t)\,dt$ and $f(x)$ must differ by a constant. Equation (6) has been obtained in some way from the Riemann definition of the definite integral. However, at this point students are still reeling from the brand new idea of varying the right endpoint of an integral, which is hard to motivate out of the blue. Deriving the much more concrete (and more useful) equation (5) from it seems opaque.

The approach in this note is of course functionally equivalent. But it is self-motivating via the very concrete interpretation of the integral as an average. It allows evaluation of definite integrals via antiderivatives earlier. It is not a free ride, since it still requires the conventional argument for equation (6) for general $g(x)$. If the instructor (or some student) wishes to deemphasize the MVT, the use of equation 3 in the proof of equation 5 can be replaced with equation 1.

We also remark that equations (3) and (5) are merely the “0th order” versions of Taylor's Theorem with the MVT and integral forms for the remainder term respectively. The approach in this note brings out this parallelism early.

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