EXPONENTIALS AND LOGARITHMS "THE OTHER WAY"

MARTIN PERGLER

The *ln* function plays three roles in one-variable calculus:

(R1) $\frac{d}{dx}b^x = (\ln b)b^x$, in particular $\ln b = \frac{d}{dx}b^x\big|_{x=0}$.
(R2) $\ln b = \int_1^b \frac{dx}{x}$; and
(R3) $\ln b = \log_e b$, the logarithm with base "some crazy number" e.

"Traditional" elementary calculus texts, such as [1, 7, 8], define ln via (R2), and then use it to define the number e, the function e^x, and finally the more general exponential functions b^x for different bases b. The roles (R1) and (R3) become computational facts. The main reason for this roundabout approach is the difficulty of extending the definition of b^x from $x \in \mathbf{Q}$ to $x \in \mathbf{R}$.

Some "reform" calculus texts, such as [3, 4], sidestep the extension issue altogether. They introduce ln via (R3), and then observe its reapparance in (R1) and (R2), thus justifying its importance. This is less roundabout, but foundationally incomplete.

Can one introduce ln via (R1) in a foundationally complete manner? We show the answer is yes. Extending functions from $\mathbf{Q}$ to $\mathbf{R}$ requires uniform continuity; apart from this, only the definition of the derivative is necessary to define ln via (R1) and to show (R3) is then satisfied. Some proofs can be simplified by use of the Mean Value Theorem. Integration is only required to show ln thus defined also satisfies (R2).

While we briefly discuss some possible classroom approaches at the end, this note is not intended for classroom use. In particular, many details one would need to present in an elementary calculus class are omitted here, and many details presented below are best sidestepped there. Few of the steps are truly new, but I don't know of any other attempt to put everything together in this manner.

1. RATIONAL EXPONENTIALS AND LOGARITHMS

Proposition-Definition 1 (Rational exponentials)**.** *For a fixed $b > 0$, the formula $E_b(p/q) = b^{p/q} = \sqrt[q]{b^p}$ defines a function $E_b : \mathbf{Q} \to \mathbf{R}^+$ satisfying the* fundamental relation $E_b(x+y) = E_b(x)E_b(y)$. *The function E_b is the* rational exponential with base b, *and $E_b(1) = b$.*

The fundamental relation is enough to recover the other standard rules for powers. For instance, $E_b(0) = 1$ since $E_b(x+0) = E_b(x)E_b(0)$; and $E(xy) = E(x)^y$, $(y = p/q)$, first by induction on p when $q = 1$, and then by raising both sides to the qth power. It also follows that E_b is monotonic (unless $b = 1$) and hence invertible.

Proposition-Definition 2 (Rational logarithms)**.** *For a fixed $b > 0$, $b \neq 1$, the formula $L_b(x) = \log_b x = E_b^{-1}(x)$ defines a function L_b* : Range$(E_b) \to \mathbf{Q}$ *satisfying the* fundamental relation $L_b(xy) = L_b(x) + L_b(y)$. *The function L_b is the* logarithm with base b *and* $L_b(b) = 1$.

The fundamental relation implies the other standard properties of logarithms.

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2. Irrational exponents and continuous extensions

A sketch-graph of $E_b : \mathbf{Q} \to \mathbf{R}^+$ (for fixed b) consists of an infinitely fine mesh of points coalescing into a line, i.e. E_b is continuous as a function of $\mathbf{Q}$. This suggests defining b^x for all real x by just "connecting the dots". Indeed, the following seems altogether too believable.

Theorem 3 (FALSE)**.** *Suppose Q is a dense subset of a set R, and let $f : Q \to \mathbf{R}$ be a continuous function. Then there is a unique continuous function $\tilde{f} : R \to \mathbf{R}$ which extends f, i.e. $\tilde{f}$ restricted to Q is the same as f.*

To find $\tilde{f}(x)$, we approximate x by elements $x' \in Q$ and set $\tilde{f}(x) = \lim_{x' \to x} f(x')$. The continuity of f should somehow imply that this limit exists and is unique. However, consider any function $g : \mathbf{R} \to \mathbf{R}$ which is continuous except for an essential discontinuity at an irrational number ξ, and let $f = g|\mathbf{Q}$. The function f is continuous, but Theorem 3 breaks down trying to define $\tilde{f}(\xi)$.

Theorem 4 (Continuous extension from a dense set)**.** *Theorem 3 becomes true if f is uniformly continuous on all bounded subsets of Q. In this case, $\tilde{f}$ is uniformly continuous on all bounded subsets of $\mathbf{R}$.*

The proof starts along the lines outlined above, but ends up involving the completeness of the codomain $\mathbf{R}$ as well as uniform continuity (see [6, Theorem 15.4], for instance.)

Even though it is "difficult", I believe this theorem should be mentioned on some level in "serious" calculus or beginning real analysis courses, because it underlies the use of technology in mathematics! Plotting a function on a computer or graphing calculator involves the machine calculating values on a fairly fine mesh of points and interpolating in between on the screen. We then further interpolate between the individual pixels with our eyes. The uselessness of technology for graphing the Dirichlet function ($D(x) = 1$ if $x \in \mathbf{Q}$ and $D(x) = 0$ if $x \in \mathbf{R} - \mathbf{Q}$), or even the function $S(x) = \sin(1/x)$, arises exactly from some "crazy" lack of continuity and the associated difficulties in approximation. To what level it is appropriate to discuss the uniformity hypothesis varies with the course and the students, of course, but the function $S(x)$ begs at least a mention of it.

We verify below that $E_b : \mathbf{Q} \to \mathbf{R}^+$ is uniformly continuous on bounded intervals. For now, assume this and define b^x for $x \in \mathbf{R}$ as $\tilde{E}_b : \mathbf{R} \to \mathbf{R}^+$, given by continuous extension (dropping the ˜). The fundamental relation extends to $\mathbf{R}$ by continuity, and shows that the extended function is also monotonic (unless $b = 1$) and unbounded, so its range is all of $\mathbf{R}^+$ and it has a continuous and monotonic inverse $L_b(x) = \log_b x$.

Theorem-Definition 5 (Exponentials and logarithms)**.** *There is a one-to-one correspondence between*

1. *Continuous functions $E : \mathbf{R} \to \mathbf{R}^+$, satisfying the fundamental relation $E(x+y) = E(x)E(y)$, called exponentials.*
2. *Nonconstant continuous functions $L : \mathbf{R}^+ \to \mathbf{R}$, satisfying the fundamental relation $L(xy) = L(x) + L(y)$, called logarithms; and*
3. *Positive real numbers b, called bases.*

The corresponding functions E and L are inverses; $E(1) = b$, $L(b) = 1$, $E(x) = b^x$, and $b^{L(x)} = x$.

Sketch-proof. If $E(x)$ is continuous and satisfies the fundamental relation (extended from $\mathbf{Q}$ to $\mathbf{R}$ as above), then $E(px/q) = E(x)^{p/q}$ as discussed after Proposition-Definition 1.

This persists for p/q replaced by any real number by continuity, and so we conclude that $E(x) = E(1)^x$. The results for logarithms follow by similar arguments and inversion. $\square$

3. UNIFORM CONTINUITY OF $E_b : \mathbf{Q} \to \mathbf{R}^+$

To show E_b is uniformly continuous on bounded subsets of $\mathbf{Q}$, and thus to complete its extension to $\mathbf{R}$, we use the formula

$$|E_b(x) - E_b(y)| = E_b(y)|E_b(x-y) - E_b(0)|.$$

Since E_b is monotonic, it is bounded on any bounded interval, and so all we need to prove is the following

Proposition 6. $\lim_{x\to 0} E_b(x) = 1$ *($x \to 0$ through* $\mathbf{Q}$*).*

We prove the case where $b > 1$ and $x \to 0^+$. The other cases follow similarly or by substituting $1/b$ for b.

First proof. By the Pinching Theorem, it suffices to show $1 \le b^x \le 1 + xb$ for $0 < x < 1$. Fix x and consider the function $g(b) = 1 + bx - b^x$. Now, $g'(b) = x(1 - b^{x-1}) > 0$, so $g(b)$ is increasing. Since $g(1) = x > 0$, this means $g(b) > 0$ for all $b \ge 1$. $\square$

In the above proof, we use the power rule only for rational exponents. However, we also use a consequence of the Mean Value Theorem (or related ideas), namely that a function with positive derivative on an interval is increasing there. With a bit more effort, we can avoid this.

Lemma 7. *If $b > 1$ and n is a positive integer, then $1 \le b^{1/n} \le 1 + b/n$.*

Proof. This first inequality is clear. For the second, assume $b^{1/n} > 1 + b/n$. Then $b > (1 + b/n)^n > 1 + nb/n = b + 1$, a contradiction. $\square$

Second proof of Proposition 6. By Lemma 7 and the Pinching Theorem, $E_b(1/n) \to 1$. So the limit is 1 as $x \to 0^+$, since the values of E_b evaluated in between the points $\{1/n\}$ are constrained by monotonicity.

We can make the last sentence more explicit: find the integer n such that $1/(n+1) \le x < 1/n$. Then $n + 1 \ge 1/x$, and so $n \ge 1/x - 1 = (1-x)/x > 0$. Hence, using Lemma 7 and monotonicity, we obtain an alternate pinching inequality

$$1 \le b^x \le b^{1/n} \le 1 + b/n \le 1 + b\frac{x}{1-x} \tag{1}$$

Here is yet another approach via geometric series. From Lemma 7, $1 \le b^{1/q} \le 1 + b/q$, so that $1 \le b^{p/q} \le (1 + b/q)^p$. Expand the final expression using the Binomial Theorem, noting that $\binom{p}{k} \le p^k$, so that $\binom{p}{k}(b/q)^k \le b^k(p/q)^k \le b(p/q)^k$. Thus we get a finite subseries of the infinite geometric series with first term $b(p/q)$ and ratio p/q. Summing this series recovers the equation (1). $\square$

4. THE DERIVATIVE OF b^x

Theorem 8. *There is function $\lambda : \mathbf{R}^+ \to \mathbf{R}$ such that for all $b > 0$, $\frac{d}{dx}b^x = \lambda(b)b^x$. In particular, each E_b is a differentiable function.*

For the moment, we won't use the name "ln" for λ, until we show later that it is a logarithm. For notational elegance, we can define $\lambda(b) = \frac{d}{dx}b^x\big|_{x=0}$. We will need the following

Lemma 9. *Suppose $\alpha > 0$ and $z > 1$. Then $(1+\alpha)^z \ge 1 + \alpha z$.*

First Proof, requiring MVT. Let $g(\alpha) = (1+\alpha)^z - (1+\alpha z)$. We have $g'(\alpha) = z(1+\alpha)^{z-1} - z > 0$ and so g is increasing (MVT!) for $\alpha > 0$. Since $g(0) = 0$, this implies the Lemma. (By continuity, we may restrict to $z \in \mathbf{Q}$.) □

Second Proof, no MVT but messier. Suppose $z \in \mathbf{Q}$, so $z = p/q$, $p > q$. The Lemma is equivalent to the relation

$$(1+\alpha)^p \geq (1+\alpha p/q)^q \tag{2}$$

Expand both sides using the Binomial Theorem, and call L_k and R_k the coefficients of α^k on the left hand and right hand sides respectively. Then

$$L_k = \binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!} \tag{3}$$

$$R_k = \binom{q}{k}(p/q)^k = \frac{q(q-1)\dots(q-k+1)}{k!}\left(\frac{p}{q}\right)^k. \tag{4}$$

However, since $p > q$, $(p/q)(q-i) \leq (p-i)$ for all $0 \leq i < q$ and so $L_k \geq R_k$ for $0 \leq k \leq p < q$. Also, $L_k > 0 = R_k$ for $p < k \leq q$. Since $\alpha > 0$, this implies relation (2). □

Proof of Theorem 8. The difference quotient for computing $E_b'(x)$ is

$$\frac{b^{x+h} - b^x}{h} = \frac{b^h - 1}{h} b^x$$

so it suffices to show that the function $F(x,h) = (x^h - 1)/h$, defined for $h \neq 0$, tends to a limit as $h \to 0$. Graphing $F(x,h)$ for various h strongly suggests that this is the case, and that we should be able to prove it by some sort of pinching. An obvious idea is to calculate $F(x,h) - F(x,k)$, for small positive h and small negative k, but this is a mess. Instead, we remark that

$$F(x,-h) = \frac{x^{-h} - 1}{-h} = x^{-h}\frac{1 - x^h}{-h} = x^{-h}F(x,h), \qquad \text{and} \tag{5}$$

$$F(x,kh) = \frac{x^{kh} - 1}{kh} = \frac{1}{k}F(x^k,h). \tag{6}$$

Suppose now that $h > 1$ and $x > 1$. Lemma 9 implies that

$$F(x,h) = \frac{(1+(x-1))^h - 1}{h} \geq \frac{1 + h(x-1) - 1}{h} = x - 1. \tag{7}$$

Applying this to the right hand side of (6) (with $k > 0$), we obtain

$$F(x,kh) \geq (1/k)(x^k - 1) = F(x,k) \tag{8}$$

which implies $F(x,k)$ increases as a function of k. Chasing through the sign changes and applying (5) as required, we discover that this remains true for $0 < x < 1$ and regardless of the sign of k.

Finally, letting $h \to 0$ in (5), we see $F(x,h)/F(x,-h) \to 1$. Since $F(x,h)$ increases as a function of h, this implies $F(x,h)$ is pinched to a limit $\lambda(x)$ as $h \to 0$. □

This approach to differentiating $E_b(x) = b^x$ is similar to the standard proof that $\frac{d}{dx}\sin x = \cos x$. There one uses trigonometric identities to reduce to the evaluation of the limits $\lim_{h\to 0}\frac{\sin h}{h}$ and $\lim_{h\to 0}\frac{1-\cos h}{h}$, which are independent of x. The reduction in the b^x case to the limit $\lim_{h\to 0}\frac{b^h-1}{h}$ is trivial. However, this limit is now a “strange” function of b, and proving this limit exists without yet being able to really get our hands on the function values presents the main challenge.

5. THE ANTIDERIVATIVE OF x^{-1}

Observe that $F(x,h) = \int_1^x t^{h-1}\,dt$. If we set $h = 0$, $F(x,h)$ is no longer defined, but $\int_1^x t^{-1}\,dt$ must still be a well-defined function of x by the Fundamental Theorem of Calculus.

Theorem 10. $\lambda(x) = \int_1^x t^{-1}\,dt$ *and so* $\frac{d}{dx}\lambda(x) = 1/x$.

Proof. The obvious idea is to just write

$$\int_1^x t^{-1}\,dt = \int_1^x \big(\lim_{h\to 0} t^{h-1}\big)\,dt = \lim_{h\to 0}\int_1^x t^{h-1}\,dt = \lim_{h\to 0} F(x,h) = \lambda(x).$$

This involves interchanging the limit and integral operations, which requires uniform convergence. Alternatively, let $\Lambda(x) = \int_1^x t^{-1}\,dt$. We claim $\Lambda(x) = \lambda(x)$. Suppose $x > 1$. If $h > 0$, then $x^{-1+h} < x^{-1} < x^{-1+h}$ so $F(x,-h) < \Lambda(x) < F(x,h)$ after integration. But $\lambda(x)$ is the only function which satisfies this as $h \to 0$. □

6. $\lambda(x)$ AS A LOGARITHM

We now have three different but compatible interpretations (or definitions) of $\lambda(x)$. The first two are the roles (R1) and (R2) stated in the introduction. The third is the "limit" definition involving the ratio $F(x,h)$, which which actually underlies both of the others. We can use any of these to prove the following

Proposition 11. *The function* $\lambda(x)$ *has the following properties:*

1. $\lambda(1) = 0$, $\lambda(x) > 0$ *for* $x > 1$, *and* $\lambda(x) < 0$ *for* $x < 1$.
2. $\lambda(xy) = \lambda(x) + \lambda(y)$.
3. λ *is an unbounded increasing continuous function.*

and thus is a logarithm.

It suffices to prove properties 1 and 2, and the continuity part of property 3. The increasing and unbounded part of 3 is then automatic, since for $y > 1$ we get $\lambda(xy) = \lambda(x) + \lambda(y) > \lambda(x)$ and $\lambda(x^n) = n\lambda(x)$, and λ is not identically 1.

"Limit" proof. For property 1, suppose $x > 1$. If $h > 0$, then $x^h > 1$, so $hF(x,h) = x^h - 1 > 0$ and thus $F(x,h) > 0$. If $h < 0$, then $x^h < 1$, so $hF(x,h) < 0$ and $F(x,h) > 0$ as before. Thus the limit function $\lambda(x)$ is positive. The case $x < 1$ is similar.

For property 2, let $h \to 0$ in the following identity:

$$F(xy,h) = \frac{x^h(y^h - 1) + x^h - 1}{h} = x^h F(y,h) + F(x,h). \tag{9}$$

Finally, to see λ is continuous, it suffices to check that $\lambda(xy) - \lambda(x) = \lambda(y)$ can be made arbitrarily small for y close to 1. But this follows from fixing some small $h > 0$ in the pinching inequality $F(y,-h) \le \lambda(y) \le F(y,h)$. □

"Derivative" proof. Property 1 is immediate. For property 2, calculate $\frac{d}{dz}\big|_{z=0}(xy)^z$ in two ways. On the one hand, it equals $\lambda(xy)(xy)^z\big|_{z=0} = \lambda(xy)$. On the other hand, writing $(xy)^z = x^z y^z$ and using the product rule, it equals $\lambda(x) + \lambda(y)$. To prove continuity, use the limit definition of the derivative and proceed as in the previous proof. □

"Integral" proof. Property 1 and continuity follow directly from basic properties of the integral and the Fundamental Theorem of Calculus. Property 2 follows from the standard calculation $\frac{d}{dx}\lambda(xy) = \lambda'(xy)y = y/(xy) = 1/x$. Since $\lambda(x)$ is itself an antiderivative of $1/x$,

$\lambda(xy)-\lambda(x)$ is independent of x, i.e. $\lambda(xy)=\lambda(x)+f(y)$ for some function $f(y)$. Finally $\lambda(1y)=\lambda(1)+f(y)=f(y)$, so $f(y)=\lambda(y)$. □

Since $\ln x=\lambda(x)$ is a logarithm, there is necessarily some unique base e such that $\ln e=1$. Furthermore, we can use the "integral" definition and the change of base formula to differentiate logarithms with any base. Indeed, all logarithms and exponentials must not only be continuous (as in Theorem-Definition 5) but also differentiable.

Some traditional calculus texts, such as [1, 7], prepare the shock of the "integral" definition of ln by first defining logarithms as as *differentiable* functions satisfying the fundamental relation $L(xy)=L(x)+L(y)$. Then it is shown by change of variable that $L'(x)=L'(1)/x$ and so ln is "natural" in that $L'(1)=1$. Differentiability as a requirement, however, is somewhat unnatural in what is otherwise a calculus-free concept.

7. The number e

In our approach, the base e is of course still *natural*, in that it is the base the logarithm $\ln=\lambda$ happens to have. It thus makes other calculus formulas take on their easiest form, so much so that for instance the formula for $\frac{d}{dx}b^x$ for general b can now be safely forgotten, its use replaced by the chain rule. However, the base e is not *privileged*, in that it has no foundational role validating the use of other bases.

We can push the analogy with trigonometric functions further, and say "e is the natural parameter (=base) which solves the differential equation $y'=y$" just as "π is the natural parameter (=angle measure) which solves the differential equation $y''=-y$".

The limit formulas $e=\lim_{x\to 0}(1+x)^{1/x}=\lim_{x\to\infty}(1+1/x)^x$, used to define "the crazy number" e in the third role (R3) of ln, can of course be obtained using the standard argument of computing $\lim_{x\to 0}(\ln(1+x)-\ln 1)/x$ in 2 ways; or via the following plausibility argument ([3, Section 4.3]): Since $1=\ln e=\lim_{x\to 0}(e^x-1)/x$, for x small we have $e^x-1\approx x$, hence $e\approx(1+x)^{1/x}$.

8. Classroom use

This note arose from accumulated frustration. As an undergraduate taking "traditional" elementary calculus, I felt that the definition of b^x via the "integral" definition of ln was an unnatural conjuring trick—a feeling shared by many of my friends who are not professional mathematicians, but have taken a "traditional" calculus course, and still remember the outline of the development—an admittedly rather selective set of qualifications! My interest was also drawn by the note [2], where the pinching of the graph of $\ln x$ by graphs of $F(x,h)$ was "explained" by computing the limit using l'Hôpital's Rule.

Later, I grew to appreciate the conciseness and elegance of the "traditional" approach, but my frustration increased when I started teaching calculus (in a "traditional" textbook environment) and saw my students' minds become tangled up in knots reconciling their precalculus notion of logarithms with the new integral definition. To them, the "traditional" approach seems even more unnecessarily complicated than it had to me, since approximation is second-nature to those raised on a diet of graphing calculators. So I became frustrated that the theoretical basis of approximation is not discussed in standard calculus courses.

Finally, my frustration reached its peak when I saw in [3] the motivationally undoubtedly "correct" approach of presenting $\frac{d}{dx}b^x$ and *e before* integration—but without any discussion of the "definition" of b^x for irrational x, and hence (to me) unsatisfyingly foundationally incomplete. Could it be made complete without too much high-powered machinery? If yes, then omission of some or even all of the details becomes a purely paedagogical decision, along the same lines of one deciding which lemmas and calculations to skip or merely glance over in reading a research paper. If no, then following that path is an *exploration* rather than a *development* of calculus— something philosophically quite different, even if either can be fully appropriate depending on circumstances.

As this note indicates, it *is* possible to follow this approach quite rigorously, but the amount of detail would be mind-numbing to any student struggling with understanding calculus. However, the development falls naturally into modular segments

- (S1) (a) Properties of b^x for $x \in \mathbf{Q}$
  - (b) (Uniform) continuity of b^x for $x \in \mathbf{Q}$
  - (c) Extension to $x \in \mathbf{R}$
- (S2) (a) Reduction of $\frac{d}{dx}b^x$ to $F(x,h)$
  - (b) $F(x,h)$ increases with h
  - (c) $F(x,h)$ is pinched to a limit.
- (S3) $\ln x$ as antiderivative
- (S4) $\ln x$ as logarithm
- (S5) The role of e

Each segment can be covered in detail, merely briefly mentioned, or developed via a sequence of guided exercises, largely independent of the treatment of the other segments. Also (S4) and (S5) can at least partially precede (S3), which may take place much later, after integration is covered.

In my moderately theoretical first-year calculus course at the University of Chicago, based on [1], I largely gloss over the uniformity issues in (S1). I don't prove (S2)(b), which students find much more believable than the $F(x,h)/F(x,-h)$ calculation in (S2)(c). I've tried placing (S1),(S2), (S4) both before and after covering the Mean Value Theorem and before integration. Classroom notes are available [5]. My students' minds still get tangled up in knots, but in different ones. Progress?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO IL 60637
E-mail address: pergler@math.uchicago.edu