



## More on the Series for In 2

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and

$$(n^2 + 4n)^2 + (4n + 8)^2 = (n^2 + 4n + 8)^2,$$

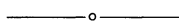
so that  $x, y$  and  $\sqrt{x^2 + y^2}$  form a right triangle for each natural number  $n$ . To prove that  $x, y$  in (5) are relatively prime, assume that they have a common factor  $p$ . Then

$$4n + 1 = ip \quad \text{and} \quad 8n^2 + 4n = jp$$

for integers  $i, j$ . Multiplying the first equation by  $2n$  and subtracting from the second, we have  $2n = mp$  ( $m$ , integral). From this and the first equation, we get  $kp = 1$  ( $k$ , integral) and hence  $p = 1$ . Finally, we verify that  $x, y$  in (6) are relatively prime when  $n$  is odd. Assume that

$$n^2 + 4n = ip \quad \text{and} \quad 4n + 8 = jp$$

for integers  $i, j$ . Since  $n^2 + 4n$  is odd,  $p$  must be odd and  $j$  must be a multiple of 4. Hence,  $n + 2 = mp$  (integral  $m$ ). Subtracting  $n^2 + 2n = mnp$  from  $n^2 + 4n = ip$ , we get  $2n = kp$  (integral  $k$ ). Since  $p$  must divide  $n$ , assume that  $n = pq$  for some integer  $q$ . Then, subtracting  $n = pq$  from  $n + 2 = mp$ , we obtain  $2 = sp$  (integral  $s$ ). Hence,  $p = 1$ . This completes the proof of our theorem.



### More on the Series for $\ln 2$

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Norman Schaumberger [CMJ 18 (May 1987) 223–225] derived the series

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \tag{1}$$

directly from the inequality

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{1+k} \tag{2}$$

and extended the method to obtain series for  $\ln n$  for all  $n$ .

Here is a variant procedure. In place of (2), our point of departure is the existence of Euler's constant. For completeness, we first give a simple geometric derivation of this constant; the only sophisticated step is that a bounded monotone sequence converges.

We define

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \tag{3}$$

and

$$\gamma_n = H_n - \ln n. \tag{4}$$

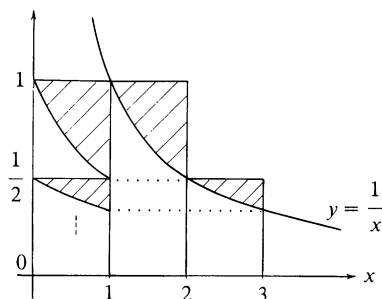
It is also convenient to put

$$\gamma'_n = H_{n-1} - \ln n. \tag{5}$$

Thus,

$$\gamma_n = \gamma'_n + \frac{1}{n}. \tag{6}$$

By way of illustration, we see below that  $\gamma'_3$  is the sum of the areas of the shaded regions above the graph of  $1/x$ . (For clarity, the axes use different scales.)



When we slide the shaded regions onto the square  $[0, 1] \times [0, 1]$ , it becomes evident that

$$\gamma'_2 < \gamma'_3 < \cdots < 1.$$

It follows that  $\gamma'_n$  approaches a limit, which we will call  $\gamma$ . In view of (6),  $\gamma_n \rightarrow \gamma$  as well. This, in fact, is the definition of Euler's constant  $\gamma$ .

Using (4) and (3), we get

$$\begin{aligned} \ln 2 &= \ln 2n - \ln n \\ &= H_{2n} - H_n - (\gamma_{2n} - \gamma_n) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - (\gamma_{2n} - \gamma_n) \\ &= 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} - (\gamma_{2n} - \gamma_n), \end{aligned}$$

and the result (1) follows, since  $\gamma_{2n} - \gamma_n \rightarrow \gamma - \gamma = 0$ .

This proof appears in Gillman and McDowell's *Calculus* [Norton, New York, 1987, p. 803]. Note that it considers only those partial sums with an even number of terms (as does Schaumberger's capsule); because the  $n$ th term approaches 0, that is sufficient.

The general case goes the same way. For example,

$$\begin{aligned} \ln 3 &= \ln 3n - \ln n \\ &= H_{3n} - H_n - (\gamma_{3n} - \gamma_n) \\ &= \sum_{k=1}^{3n} \frac{1}{k} - 3 \sum_{l=1}^n \frac{1}{3l} - (\gamma_{3n} - \gamma_n), \end{aligned}$$

and we obtain

$$\ln 3 = \left(1 + \frac{1}{2}\right) - \frac{2}{3} + \left(\frac{1}{4} + \frac{1}{5}\right) - \frac{2}{6} + \cdots.$$

*Remark.* Mark Finkelstein has come up with a very clever geometric construction of the series for  $\ln 2$  [*Amer. Math. Monthly* (June–July 1987) 541–542]. The same idea works for  $\ln n$ . To get Schaumberger's series for  $n > 2$ , divide the base interval  $[1, n]$  into  $n - 1$  equal parts and divide all further subintervals into  $n$  equal parts. The series for  $\ln 3$ , for example, is produced in the form

$$\ln 3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots.$$