and

$$(n^2+4n)^2+(4n+8)^2=(n^2+4n+8)^2,$$

so that $x$, $y$ and $\sqrt{x^2+y^2}$ form a right triangle for each natural number $n$. To prove that $x$, $y$ in (5) are relatively prime, assume that they have a common factor $p$. Then

$$4n+1=ip \quad \text{and} \quad 8n^2+4n=jp$$

for integers $i$, $j$. Multiplying the first equation by $2n$ and subtracting from the second, we have $2n=mp$ ($m$, integral). From this and the first equation, we get $kp=1$ ($k$, integral) and hence $p=1$. Finally, we verify that $x$, $y$ in (6) are relatively prime when $n$ is odd. Assume that

$$n^2+4n=ip \quad \text{and} \quad 4n+8=jp$$

for integers $i$, $j$. Since $n^2+4n$ is odd, $p$ must be odd and $j$ must be a multiple of 4. Hence, $n+2=mp$ (integral $m$). Subtracting $n^2+2n=mnp$ from $n^2+4n=ip$, we get $2n=kp$ (integral $k$). Since $p$ must divide $n$, assume that $n=pq$ for some integer $q$. Then, subtracting $n=pq$ from $n+2=mp$, we obtain $2=sp$ (integral $s$). Hence, $p=1$. This completes the proof of our theorem.

---

**More on the Series for ln 2**

Leonard Gillman, The University of Texas at Austin, Austin, TX

Norman Schaumberger [CMJ 18 (May 1987) 223–225] derived the series

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \tag{1}$$

directly from the inequality

$$\left(1+\frac{1}{k}\right)^k<e<\left(1+\frac{1}{k}\right)^{1+k} \tag{2}$$

and extended the method to obtain series for $\ln n$ for all $n$.

Here is a variant procedure. In place of (2), our point of departure is the existence of Euler's constant. For completeness, we first give a simple geometric derivation of this constant; the only sophisticated step is that a bounded monotone sequence converges.

We define

$$H_n=1+\frac{1}{2}+\cdots+\frac{1}{n} \tag{3}$$

and

$$\gamma_n=H_n-\ln n. \tag{4}$$

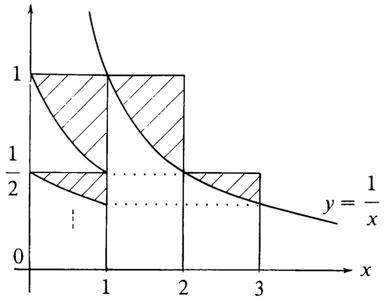
It is also convenient to put

$$\gamma_n'=H_{n-1}-\ln n. \tag{5}$$

Thus,

$$\gamma_n=\gamma_n'+\frac{1}{n}. \tag{6}$$

By way of illustration, we see below that $\gamma_3'$ is the sum of the areas of the shaded regions above the graph of $1/x$. (For clarity, the axes use different scales.)




When we slide the shaded regions onto the square $[0,1]\times[0,1]$, it becomes evident that

$$\gamma_2' < \gamma_3' < \cdots < 1.$$

It follows that $\gamma_n'$ approaches a limit, which we will call $\gamma$. In view of (6), $\gamma_n \to \gamma$ as well. This, in fact, is the definition of Euler's constant $\gamma$.

Using (4) and (3), we get

$$\begin{aligned}\ln 2 &= \ln 2n - \ln n \\ &= H_{2n} - H_n - (\gamma_{2n} - \gamma_n) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - (\gamma_{2n} - \gamma_n) \\ &= 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} - (\gamma_{2n} - \gamma_n),\end{aligned}$$

and the result (1) follows, since $\gamma_{2n} - \gamma_n \to \gamma - \gamma = 0$.

This proof appears in Gillman and McDowell's *Calculus* [Norton, New York, 1987, p. 803]. Note that it considers only those partial sums with an even number of terms (as does Schaumberger's capsule); because the $n$th term approaches 0, that is sufficient.

The general case goes the same way. For example,

$$\begin{aligned}\ln 3 &= \ln 3n - \ln n \\ &= H_{3n} - H_n - (\gamma_{3n} - \gamma_n) \\ &= \sum_{k=1}^{3n} \frac{1}{k} - 3\sum_{l=1}^{n} \frac{1}{3l} - (\gamma_{3n} - \gamma_n),\end{aligned}$$

and we obtain

$$\ln 3 = \left(1 + \frac{1}{2}\right) - \frac{2}{3} + \left(\frac{1}{4} + \frac{1}{5}\right) - \frac{2}{6} + \cdots.$$

*Remark.* Mark Finkelstein has come up with a very clever geometric construction of the series for ln 2 [*Amer. Math. Monthly* (June–July 1987) 541–542]. The same idea works for ln $n$. To get Schaumberger's series for $n > 2$, divide the base interval $[1, n]$ into $n-1$ equal parts and divide all further subintervals into $n$ equal parts. The series for ln 3, for example, is produced in the form

$$\ln 3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots.$$