WOMP: INTRO TO VECTOR BUNDLES

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Contents

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1. Introduction and definition

Vector bundles are a key technical tool in geometry and topology. The key examples are the tangent and cotangent bundles, which formalize the intuition of "an infinitesimal tangent vector".

Things you can usefully study in this context:

- Understanding a particular vector field or differential form
- Understanding a particular vector bundle, especially the tangent bundle, especially via characteristic classes
- Understanding a manifold (or space) by the vector bundles on it (K-theory)
- Classifying manifolds via "what putative tangent bundles can a space admit?" (surgery theory)

1.1. Physical intuition.

The tangent bundle of a manifold is the phase space of "a particle with velocity": a point in the tangent bundle gives a position in the manifold and a velocity.

More subtly perhaps, the cotangent bundle of a manifold is the phase space of "a particle with *momentum*".

A tangent vector field (a tangent vector for each point is the manifold) on a manifold can be interpreted as a force field, for instance if you have a distribution of electrical charge acting on a test particle, or a bumpy landscape and gravity. (These are both examples of gradient vector fields for a potential: $\xi = -\operatorname{grad} U$ for a function U, called the potential.)

1.2. Examples.

2 key examples:

- Möbius strip
- tangent bundle of sphere

The Möbius strip can be interpreted as a vector bundle over the circle: it's a "twisted product" of a line and a circle (unlike the cylinder, which is the untwisted product). Note that this is *asymmetric*: the line is twisted *as you move around the circle*, but not conversely. Incidentally, there are only 2 (classes of) vector bundles on the circle; we'll discuss more below.

For the sphere, a key observation is there aren't global directions (the bundle isn't trivial): away from the poles, you can say "North/South" and "West/East", but this breaks down at the poles and this is inevitable: there is no way to have consistent global directions on the sphere, because the tangent bundle isn't trivial (the manifold isn't *parallelizable*); contrast with the circle or torus (or S^3!). See also "combing the sphere (the impossibility thereof)".

1.3. Definition.

A vector bundle is a special case of a fiber bundle.

Intuitively, a fiber bundle is a twisted product. The formal definition is somewhat involved.

Formally, a fiber bundle is a map $\pi \colon E \to B$ which is "locally modeled on the projection from a product"; local *in the base*, meaning that for every point $b \in B$, there is a neighborhood $U \subset B$ such that $\pi_U \colon \pi^{-1}U \to U$ is fiberwise isomorphic to the "projection from a product" map $F \times U \to U$.

This is analogous to the (topological) definition of a manifold (each point has neighborhood homeomorphic to (open subset of) $\mathbf{R}^n$).

One can also define a fiber bundle with a structure group: you require the patchings to be compatible (the fiberwise maps are in some group G); this is analogous to the differential definition of a manifold (with charts and atlases).

A vector field is simply a fiber bundle with fiber a vector space V and structure group $\mathrm{GL}(V)$ (if you prefer, $\mathbf{R}^n$ and $\mathrm{GL}(n)$).

A section of a fiber bundle is a map $s\colon B \to E$ that is right inverse to the projection: $\pi s = \mathrm{id}_B$. In other words, assigning to each point in the base a point in the fiber *over that point.* In the case of vector bundles, we call a section a *vector field.*

1.3.1. *Categorical notes.* We can define a category of fiber bundles or vector bundles *on a given space*, and define maps between bundles (fiberwise maps), subbundles, quotient bundles, products, coproducts, etc.

You can also have maps between fiber bundles on different spaces; intuitively, you have a map of the base spaces and of the total spaces, which is a good map fiberwise. Formally, you pull back the bundle from the target to the source, and have a map of fiber bundles, both over the source.

Note that you obviously can't sum fiber bundles over different spaces, for instance.

If you want, you can call "fiber bundles with fiber F over a space X" a contrafunctor from (spaces and continuous maps) to (categories and functors): a map between spaces induces a functor between the categories, via pulling back bundles.

For vector bundles, you can do all the usual linear algebra constructions, which we'll detail below.

1.4. **Pulling bundles back.** Fiber bundles naturally pull back: given a map $f\colon B' \to B$ and a fiber bundle over B ($\pi\colon E \to B$), the fiber product (pull back) $B' \times_{f,\pi} E$ is a fiber bundle over B'.

In some cases you can push bundles forward; this is a "wrong way" map (aka, shriek map, Umkehr map, Gysin map; denoted $f_!$. Algebraic geometers love these.).

1.5. **Constructing the tangent bundle.** One way to construct the tangent bundle to a manifold is to present the manifold via an atlas, and pull back the tangent bundle on $\mathbf{R}^n$ and patch.

Another way is to embed (or just immerse!) the manifold in $\mathbf{R}^{n+k}$ and define the subbundle of "vectors tangent to the submanifold".

1.6. **Normal bundle.** Given an embedded (or immersed!) manifold $N \to M$, we can define the normal bundle on N. If M is Riemannian, then it's just $(TN)^\perp \subset TM|_N$ (the perpendicular of the tangent bundle, restricted to N). If there's no Riemannian structure, then the normal bundle is instead a quotient: $TM|_N/TN$.

Note that a normal bundle is *not intrinsic* to a manifold: it comes from an embedding, say in $\mathbf{R}^N$. However, there is an intrinsic *stable normal bundle* (due to Spivak): embeddings into a big enough $\mathbf{R}^N$ are all homotopic (take $N = 2n+1$ or so and count dimensions of a homotopy. Better (for this purpose), look at connectivity of maps $O(n) \to O := O(\infty)$.), so the normal bundle is well-defined (up to adding on trivial bundles for embedding in a bigger $\mathbf{R}^N$). (Yes, you can just embed in Hilbert space to be canonical.)

In fact, the stable normal bundle generalizes better to other topological settings (PL-manifolds, topological manifolds) than the tangent bundle: take an embedding, take a neighborhood, and you have a normal bundle – only defined stably (sweeping technicalities under the rug, this works).

Yes, there is a conormal bundle. No, I've never used it.

1.7. **Algebraic intuition.** Fiber bundles are twisted products, in the sense of homological algebra (or at least some topological analog), and one generally writes them as $F \to E \to B$.

Recall that algebraically, given $\mathbf{Z}/10$ and $\mathbf{Z}/10$, there is a product short exact sequence:

$$1 \to \mathbf{Z}/10 \to \mathbf{Z}/10 \times \mathbf{Z}/10 \to \mathbf{Z}/10 \to 1$$

. . . but there are also twisted products, like:

$$1 \to \mathbf{Z}/10 \to \mathbf{Z}/100 \to \mathbf{Z}/10 \to 1$$

(which you're familiar with from "addition with carrying"!)

In the same way, fiber bundles are topological twisted products.

You also might write them as:

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

. . . especially if you're thinking of computing the (co)homology of E via a spectral sequence (say, the Leray-Serre spectral sequence), so you put the (co)homology of F and B along the axes, and get the (co)homology of E in the first quadrant.

. . . or if you just find this picture more suggestive of "fibers twisting over a base".

1.8. **Infinitesimal geometry.** Geometrically, the key use is to define geometry infinitesimally: a geometric structure on a space is an algebraic structure on the tangent bundle. The most familiar example is a Riemannian structure on a manifold, which is an inner product on each tangent space (varying smoothly). This is the Cartan notion of geometry. The historical generalizations are as follows:

In Euclidean geometry, you consider the space $X = \mathbf{R}^2$, with the group $G = \mathrm{Aff}(O(2))$ of rigid affine motions of the plane: two figures (triangles, say) are congruent if and only if they are in the same orbit of this group.

Klein generalized this to any Lie group G, with a closed subgroup H (such that $X = G/H$ is connected, i.e., $H \to G$ is 0-connected); for the Euclidean plane, $H = O(2)$. This now includes the familiar spherical and hyperbolic geometries. Note that these are homogeneous spaces: every point is the same (G acts transitively on G/H, obviously, and this is why every homogeneous space X is G/H: take $G = \mathrm{Isom}(X)$, and $H = \mathrm{Stab}(x)$ for some base point). See Klein's "Erlangen Program" for further details.

To also include spaces that are not homogeneous (say, a dented sphere, or your favorite torus embedded in $\mathbf{R}^3$), the curvature needs to change from point to point, which is a Cartan geometry: it is locally modeled on a Klein geometry, just as manifolds are (topologically) modeled on $\mathbf{R}^n$ and vector bundles are locally modeled on a projection from a product.

1.9. **Motivation.** An overarching theme is "make objects vary in a moduli". Thus instead of just studying a vector field, you study the space of all vector fields (which is sections of a vector bundle), and rather than just studying the tangent bundle, you study vector bundles generally, and see the tangent bundle as a particular vector bundle.

1.10. **Canonical structure on tangent and cotangent bundle.** The tangent bundle of a vector bundle on a manifold has a special structure: since the vector bundle is locally a product, the tangent space at each point is a product (a direct sum), and we call the pieces the *horizontal* direction (tangent to the base) and the *vertical* direction (tangent to the fiber). Indeed, the whole tangent bundle decomposes as a sum of a horizontal subbundle and vertical subbundle.

The tangent bundle *of* the tangent bundle $T(TM)$ is rather special: the horizontal and vertical subbundle are canonically isomorphic (they're both isomorphic to TM, or rather, the pullback of TM under $TM \to M$), so it carries an *almost complex structure* (which is what you say if $V = W \oplus W$).

Similarly, the tangent bundle *of* the cotangent bundle carries a symplectic structure: the horizontal component is isomorphic to (the pullback of) TM, while the horizontal component is isomorphic to (the pullback of) T^*M, and you can pair these against each other and do Hamiltonian mechanics.

You can do Lagrangian mechanics on the tangent bundle, and Hamiltonian mechanics on the cotangent bundle; they are related via the Legendre transform. I'm told this also leads to *mirror symmetry*, but I don't know the story.

Beware that it's easy to confuse all these tangent and cotangent vectors, and indeed in the Atiyah-Singer index theorem paper I'm told that they are confused.

2. Structures on a vector bundle: vector fields and differential forms

This isn't the focus; a few points on this often skipped area.

(Local) classification of vector fields:

- Picard's theorem on local solution (and uniqueness) of ODEs (integrability of non-vanishing vector fields): this really says that every non-vanishing vector field is ∂_x in disguise ("rectifiability": you can "straighten 'em out"), and this is obviously solveable (by one variable integration)
- Lie bracket of vector fields is obstruction to simultaneous integrability of two vector fields (by Clairhaut's theorem on equality of mixed partials, $[\partial_x, \partial_y] = 0$): so if you're ∂_x, ∂_y in some other coordinates, you better still commute; I think it's the only obstruction, and I think this holds for any number of vector fields (if they all commute, they're simultaneously rectifiable)

For global solutions, you get into topology!

Another natural question is "when does a vector field come from a potential?", i.e., when is it a derivative? Well, the derivative of a function is actually a 1-form (you need a metric to define grad), so you actually mean "when is a 1-form the derivative of a function (aka, exact)"? To which the (obstruction-theoretic!) answer is: first it must satisfy the local condition of closed-ness, and then if it is closed, you define its cohomology class – which is the de Rham view on cohomology, and a key subject of every class on differential algebraic topology.

2.1. **Schur functors.** Schur functors aren't well-enough known; these are Sym and Alt and all the other such functors (which come for the other representations of the symmetric group).

The most important are Alt^* (aka, $\Lambda^* = \bigoplus_k \Lambda^k$); in some way these correspond to the elementary symmetric polynomials.

Note that $\Lambda^k V$ is "(linear combinations of) (weighted) k-planes in V"; so differential k-forms are "dual to" k-planes, hence their geometric import.

To learn more on this, learn about the representation theory of the symmetric group and Frobenius-Schur duality (which is a beautiful topic but not our interest here).

3. Structure of a vector bundle, and space of vector bundles

How to analyze a vector bundle?

Conveniently, vector bundles are *represented*: the set of vector bundles on X up to (vector bundle) isomorphism is the set of homotopy classes of maps $X \to KO$, where KO is a particular space (you can take it to be the infinite Grassmannian)[1].

We say that "the functor of vector bundles is representable" (it's not just some abstract functor: it's maps into some space), and it's represented by KO.

Thus to understand a vector bundle, look at the corresponding (homotopy class of) maps, call it the *classifying map* $\xi\colon X \to KO$, and study the structure of this map, using the tools of homotopy theory and algebraic topology. A key example are characteristic classes.

3.1. Characteristic classes.

Associated to a vector bundle are cohomology classes. There are assorted ways to construct them; the simplest is "compute the cohomology of KO, and pull back those classes to X".

Note that these are homological: they must vanish for a bundle to be trivial, but that's not sufficient. (A stably trivial but not trivial bundle (like TS^2) is a kind of counter-example.)

They can be constructed as polynomials in curvature (namely, (trace of) exterior powers of curvature); this is called Chern-Weil theory.

BTW, it's easier to compute the cohomology of KO at 2 (mod 2) and away from 2 (inverting 2), rather than the integral cohomology. This happens *everywhere* in algebraic topology, and this is the simplest convincing example I know.

3.2. Characteristic numbers.

You know Gauß-Bonnet (and its polyhedral analog, Descartes' theorem on total angular defect of a convex polyhedron)? It comes from this: curvature is a characteristic class, called the Euler class, and integrating it gives the Euler number. If you do the same thing (integrate over the manifold) with other combinations of characteristic classes, you get characteristic numbers – this is good, because integers are a bit more concrete than cohomology classes. In particular, you can compare numbers across different manifolds, while you can't directly compare characteristic classes.

Using the Chern-Weil construction of characteristic classes from curvature, this is a local-to-global construction, just like "winding number of a polygonal curve is sum of exterior angles (divided by $1/2\pi$)" and "winding number of a smooth curve is integral of extrinsic curvature (divided by $1/2\pi$)" (if you let curvature be singular, namely a measure, rather than a function, these are the same example).

If you're into this, also read up on the Atiyah-Singer index theorem, a similar (and deep) result.

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[1] To the specialists: I'm being sloppy about bundles versus virtual bundles (formal differences of bundles), and $KO = \mathbf{Z} \times BO$ versus BO. This is a sketch, okay?

3.3. **Embeddings and immersions.** One compelling application[2] of characteristic classes (and more generally, the normal structure, meaning the stable normal bundle) is in understanding embeddings and immersions of manifolds.

The simplest example is that the cylinder embeds in $\mathbf{R}^2$, but the Möbius strip doesn't (indeed, it doesn't even immerse in the plane). Why not? Because the Möbius strip isn't orientable, so its stable normal bundle isn't trivial, so any particular *un*stable normal bundle can't be trivial, so it better have dimension at least 1!

In more detail: a manifold is orientable (which is equivalent to the stable normal bundle being orientable) iff its first Stiefel-Whitney class w_1 vanishes. The characteristic classes of a k-dimensional bundle vanish above dimension k, and adding on trivial bundles doesn't change characteristic classes, so if a manifold is non-orientable, its stable normal bundle must be the stabilization of a bundle of dimension at least 1. Thus we've shown that non-orientable manifolds do not embed (or immerse) in codimension 0.

If you actually compute characteristic classes (which is fun), you can do the same to get embedding restrictions on the various $\mathbf{RP}^n$.

Note that this shows a limitation of the normal bundle: it doesn't distinguish between embeddings and immersions, so if one manifold immerses but doesn't embed in another, you can't just appeal to the normal bundle.

3.4. **Homotopy groups of KO.** For reference, the homotopy groups of $KO = \mathbf{Z} \times BO$ are:

$$\begin{aligned} \pi_0 KO &= \mathbf{Z} & \qquad \pi_4 KO &= \mathbf{Z} \\ \pi_1 KO &= \mathbf{Z}/2 & \qquad \pi_5 KO &= 0 \\ \pi_2 KO &= \mathbf{Z}/2 & \qquad \pi_6 KO &= 0 \\ \pi_3 KO &= 0 & \qquad \pi_7 KO &= 0 \end{aligned}$$

... and then they repeat. Indeed, $\Omega^8 KO \approx KO$; this is called (real) Bott periodicity, and is an amazing fact, today with myriad proofs. (You did notice that away from 2 this is 4-fold periodic, no? Oh, and $\Omega^4 KO = KSp$. Will wonders never cease? Oh, but you knew that: quaternions are a phenomenon at 2.)

BTW, the 8 is related to the octonions; philosophically, the octonions mean that you can't have better than 8-fold periodicity, and the lack of a division algebra of sedenions means that it's exactly 8-fold (see Hurwitz's theorem). I don't understand exactly why this is true or if it can be made rigorous, but it's a nice philosophy.

See, over the complex numbers we have $\Omega^2 KU \approx KU$ (and the groups are $\mathbf{Z}$ in even dimension and 0 in odd dimension), which corresponds to there being no composition algebras over the complexes.

3.5. **Clutching construction.** KO is the de-looping of the orthogonal group O (this is in general true of classifying spaces); hence the homotopy groups of KO are the homotopy groups of O, shifted by a dimension: $\pi_k O = \pi_{k+1} KO$. This helps one have intuition for the homotopy of KO: think about an element of $\pi_k KO$ as a (class of) (virtual) vector bundles on S^k, or as an element of $\pi_{k-1} O$. Since the lower dimensional $O(n)$ are familiar, this helps.

[2]Indeed, the example that first got me interested in them.

You can realize this correspondence concretely via “the clutching construction”: a vector bundle over S^k is trivial(izable) on the upper hemisphere (since it is contractible), and likewise on the lower hemisphere, hence the only reason that it’s non-trivial is because of the gluing map (gluing the vector bundle on the upper hemisphere, restricted to the equator, to the vector bundle on the lower hemisphere, restricted to the equator). Gluings are parametrized by $O(n)$ (properly, an affine copy of this, as there’s no identity), hence gluing along the equator corresponds to a map $S^{k-1} \to O(n)$ – so (classes of) vector bundles on the k-sphere correspond to (homotopy classes) $\pi_{k-1}O(n)$. (You need to check that homotopy of maps and equivalence of vector bundles agree, but this is obvious.)

3.6. **General comments on G-bundles.** In general G bundles have a classifying space; I don’t know the technical conditions. There’s a simplicial construction.

We get a fiber bundle $G \to EG \to BG$, with EG contractible, (and conversely, gives such a bundle $G \to E \to X$ where E is contractible, X is a classifying space for G), so via the long exact sequence on homotopy, you get $\pi_k BG = \pi_{k+1}G$.

3.7. **Two relevant examples of classifying spaces.**
$\mathbf{Z}/2 \to S^\infty \to \mathbf{RP}^\infty$
(the limit of $S^n \to \mathbf{RP}^n$)
$S^1 \to S^\infty \to \mathbf{CP}^\infty$
(the limit of $S^{2n+1} \to \mathbf{CP}^n$)

Here, let me make them more suggestive:
$O(1) \to S^\infty \to G_{\mathbf{R}}(1,\infty)$
$U(1) \to S^\infty \to G_{\mathbf{C}}(1,\infty)$
…where by $G_K(k,n)$ I mean the Grassmannian of k-planes in n-space over K.

3.8. **Intuition for homotopy groups, especially in low dimensions.**
$K(S^0)$: dim
How can a vector bundle over S^0 (two points) fail to be trivial?
Why, it could have different dimensions over each point!
This corresponds to $\pi_0 KO = \pi_{-1}O$, so it doesn’t have an orthogonal group interpretation (well, not without some massaging).

$K(S^1)$: orient or not
From the POV of vector bundles, this is whether the bundle is orientable or not; from the POV of the clutching construction, it’s whether you glued via an orientation preserving/reversing map, and corresponds to $\pi_0 O(n) = \mathbf{Z}/2$ (for $n > 0$).

$K(S^2)$: subtler: there are $\mathbf{Z}$ of them, but once you stabilize, it’s just “spin or not”
From the POV of the orthogonal group, this corresponds to $\pi_1 O(2) = \mathbf{Z}$, but $\pi_1 O(n) = \mathbf{Z}/2$ for $n \geq 3$ (recall that $SO(3) = \mathbf{RP}^2$). For the algebraic geometer in everyone, these correspond to the complex line bundles $\mathcal{O}(k)$ (or $\mathcal{O}(-k)$; I forget the convention) – of course you’re thinking of $S^2 = \mathbf{CP}^1$ with its complex structure.

The tangent bundle of the sphere is not trivial (combing the sphere), and corresponds to $2 \in \pi_1 SO(2)$, via Euler characteristic, but is stably trivial, since it embeds in $\mathbf{R}^3$ with trivial tangent bundle (so if $x + 0 = 0$, then $x = 0$, or rather if you add a trivial and you get a trivial, then you’re stably trivial), which is what we’d expect, since

$$2 \in \ker \pi_1 \colon \pi_1 SO(2) \to \pi_1 SO(3).$$

$K(S^3)$: none on S^3 (b/c $\pi_2 G = 0$: general fact of life for Lie groups. No, I can't prove it.).

$K(S^4)$: this corresponds in some way to TQFT (Topological Quantum Field Theory). No, I don't know what those are, but John Baez and Peter May do.

...and that's it – all other homotopy groups can be understood as suspensions of these, so long as you buy Bott periodicity. In other words, the only basic topological phenomena in (real) vector bundles are:

- dimension
- orientability
- spin
- TQFT
- ...and Bott periodicity

3.9. **Group cohomology.** For group theorists who have been zoning out (or the group theorist in everyone), note that group cohomology is the cohomology of the classifying space BG!

$$H^*(G, M) = H^*(BG; M)$$

More precisely, group cohomology with coefficients in a module corresponds to *sheaf* cohomology, where we're computing cohomology with coefficients in a local ring. Perhaps only topologists think this simplifies matters, but it means you can apply algebraic topology (and homological algebra) to finite group theory!

3.10. **Model of BO: infinite Grassmannian.** Anyway, BO (which you can think of as an abstract space, up to homotopy equivalence) has nice model (concrete spaces realizing this homotopy class): it's the Grassmannian of planes[3]. BSO is the oriented Grassmannian (oriented planes). Yes, this is a 2-fold cover of BO, corresponding to $SO \to O \to \mathbf{Z}/2$. Better:

$$SO \to O \to \mathbf{Z}/2 \to BSO \to BO$$

...all as point-wise fiber bundles. You can add "$\to B\mathbf{Z}/2$" (using $\mathbf{RP}^\infty$), but this last map is not a fiber bundle, using these model spaces.

Particularly clever clogs will note that $\mathbf{Z}/2 = \{\pm 1\} = O(1)$, if you want to be extra slick (and you do: $B\mathbf{Z}/2 = BO(1)$ is the Grassmannian of lines in $\mathbf{R}^\infty$, aka, $\mathbf{RP}^\infty$).

I think there are Grassmannian models for BU, BSp, and your other favorites. (Yes, there's probably a nice model for BF_4, but I don't know it.)

Another way to produce models of classifying spaces is Milnor's bar construction (which gives a simplicial model).

4. Structure of manifolds

Vector bundles are fundamental in classifying manifolds; the theory of classification of high dimension manifolds (dimension ≥ 5, and dimension 4 if you mean "topological" manifold, not "differentiable" manifold) is called "surgery theory".

Here's how it goes: given a manifold, try to find enough invariants to determine it (reconstruct it). Firstly there's the homotopy type. Next there's the normal structure (the data of the (stable) normal bundle, or equivalently (stable) tangent

[3]You could rightly ask: which Grassmannian? I mean $BO(k)$ is the Grassmannian of k-planes, but what's BO? I think you take the mapping telescope / holim.

bundle). Given a homotopy type and a normal invariant, you get another invariant, called the "surgery obstruction". These data determine the manifold.

Note that they do go in this order: a stable normal bundle structure on a space X is a map $X \to BO$, so you first need the space X (up to homotopy equivalence)! Similarly, you need the space X and the normal invariant $X \to BO$ before you can define the surgery obstruction. This "one step at a time" (need to choose earlier steps before you can define latter steps) is called "obstruction theory", and is ubiquitous in Geometry & Topology.

Mirabile dictu, these invariants form a long exact sequence, called the "surgery exact sequence", which has terms $\mathcal{S}(M) \to \mathcal{N}(M) \to \mathbf{L}(M)$ (you can remember this by the dumb mneumonic "Saturday Night Live sequence"). Note that written this way, the $\mathbf{L}(M)$ actually measures the failure of some normal invariants to be realized by manifolds; the $\mathbf{L}(M)$ on the left of SM is what you mean by "surgery obstruction". (Also, the homotopy type is restricted – it must satisfy Poincaré duality.)

There are more wonders here: $\mathbf{L}(M)$ depends only on $\pi_1(M)$, and is 4-fold periodic. (Yes, this is related to Bott periodicity. Away from 2, (algebraic) L-theory (of the trivial group) and (topological) KO-theory agree ($\mathbf{L}_k(e) = \pi_k KO$).)

This was originally discovered by Milnor in the context of exotic spheres; see Kervaire & Milnor for original write-up (which takes *a lot* of un-wrapping to relate to this discussion, because they do everything ad hoc, using special structures of spheres, and not (hardly?) mentioning Smale's h-cobordism theorem; read a modern discussion first).

For a modern write-up, and many, many more details, see Lück's "A Basic Introduction to Surgery Theory".

4.1. K-theory.

Another direction you can take is to study topological spaces by looking at the space of all vector bundles on them. This works better if you look at *stable* vector bundles, so let's do that, and write $K^i(X)$ to be the group of stable (complex) vector bundles of dimension i; write $KO^i(X)$ if you're using real vector bundle. (Technically you're formally inverting by looking at formal differences $V - W$, and the dimension (the difference of the dimensions) is called the *virtual dimension*; this is called the *Grothendieck group.*)

This has very similar properties to ordinary (singular) cohomology, essentially because you can "patch" vector bundles, so you define a *generalized cohomology theory* to be anything satisfying these axioms that cohomology and K-theory satisfy, called the *Eilenberg-Steenrod* axioms.

Of course, since (stable) vector bundles are represented by KO, this cohomology theory is "(homotopy classes of) maps into KO".

There are many, many other generalized (co)homology theories, and they are fundamental to algebraic topology.

5. References

5.1. People.

Shmuel Weinberger : for geometric and topological aspects; also see his homepage for useful notes and suggestions.
`http://www.math.uchicago.edu/~shmuel/`

Peter May : for algebro-topological aspects.

5.2. **Books.** Allan Hatcher's book-in-progress:
`http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`
See also his other books.

Wolfgang Lück's "A Basic Introduction to Surgery Theory"
`http://www.math.uni-muenster.de/u/lueck/org/staff/publications.html`

Characteristic Classes, by Milnor and Stashef

Kervaire & Milnor "Groups of homotopy spheres: I" (there is no II); reference at:
`http://en.wikipedia.org/wiki/Exotic_sphere`

Bott and Tu is a good general reference

Andrew Ranicki has *lots* of books on surgery, including the original, "Surgery on compact manifolds", by C.T.C. Wall
Ranicki's also written very extensively on algebraic perspectives on surgery, which is an elegant formulation
`http://www.maths.ed.ac.uk/~aar/books/index.htm`

5.3. **Websites.** Wikipedia, of course. If it's lacking, fix it!

John Baez, This Week's Finds in Mathematical Physics:
(lots of very interesting informal discussion, lots of intuition)
`http://www.math.ucr.edu/home/baez/TWF.html`