CHAPTER 1

Implicit and Explicit

A unifying theme of mathematical computations is working with objects, presented implicitly or explicitly[1], especially:

- converting between different presentations
- doing computations in different presentations

This is particularly pronounced in elementary multivariable calculus and linear algebra classes, as these are high in computation and low in theory.

The reason for the centrality of presentations is that to *do* a computation, you need a handle on the object, something to compute *with:* you need a presentation.

1. Implicit and explicit presentations

For concreteness and to show the ubiquity of this perspective, we describe various implicit and explicit presentation techniques, and presentations of familiar objects.

Note that an object can have multiple explicit or implicit presentations.

1.1. Calculus: subsets of $\mathbf{R}^n$.

In calculus, for subsets of $\mathbf{R}^n$,

- *explicit* presentations are parametric equations (for connected sets); in 0 dimensions (a disconnected set of points), an enumeration
- *implicit* presentations are level sets, constraints (or sublevel sets)

1.1.1. *Calculus, 0 dimensional.*

Explicit	$x = \pm 1$
Implicit	$x^2 - 1 = 0$

1.1.2. *Calculus, 1 dimensional.*

Implicit	$x^2 + y^2 - 1 = 0$
Explicit	$(\cos t, \sin t)$, or $\left(\frac{1-t^2}{1+t^2}, \frac{t}{1+t^2}\right)$

[1] We'll switch the order of "implicit" and "explicit" around according to what feels right.

1.1.3. *Linear algebra, affine subspaces.*

- *implicit* presentations are a system of equations (or a single affine linear equation in several dimensions)
- *explicit* presentations are a parametrization

Implicit	$x + y = 1$
Explicit	$(1, 0) + t(-1, 1)$

1.2. Calculus: functions.

- *explicit* presentations include algebraic combinations of known functions (say, the elementary functions) and integrals thereof; and power series
- *implicit* presentations include functional equations, or as solutions to variational problems

1.2.1. *The Exponential function.* Explicitly, it's $\sum_{i=0}^{\infty} \frac{x^n}{n!}$, or $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. Implicitly, it's the function that satisfies $\exp' = \exp$ and $\exp(0) = 1$, or $\exp(x + y) = \exp(x)\exp(y)$ and $\exp'(0) = 1$.

1.2.2. *The Normal distribution.* Explicitly, it's: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ Implicitly, it's the maximal entropy distribution with zero mean and unit variance.

1.2.3. *The Cycloid.* Explicity, it's the cycloid: $(t - \sin t, 1 - \cos t)$. Implicitly, it's the tautochrone, or the brachistochrone.

1.2.4. *The Gamma function.* Explicity, it's

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

Implicitly, it's the function extending the factorial that's log-convex.

1.3. Categorically. Categorically, explicit and implicit presentation are presenting a subobject $S \subset X$ in dual ways, either by mapping *in* to X or mapping *out* of X:

Explicitly	in	image of $S \to X$
Implicitly	out	kernel of $X \to Y$

In particular for singleton sets (in concrete categories), it's presenting an element of a set.

Dually, you can define a quotient object Q of X:

explicitly	image of $X \to Q$
implicitly	cokernel of $A \to X$ (if $Q = X/A$)

Which of these one calls "implicit" or "explicit" is unclear, as it is dual to the usual (subset) case; I've made a suggested choice (explicit if the map contains the object we are describing, implicit if it the object isn't part of the map).

This is cleanest in algebraic categories (especially abelian categories). The set-theoretic interpretation is not as clean because quotient sets are not always given by quotienting out a subobject; as an analog, you could present the equivalence relation that defines the quotient map.

By brief way of example, consider:

- Linear algebra: a quotient space (a line) K of a space V, either as a linear functional $V \to K$, or as a hyperplane (its kernel) $H < V$
- Number theory: how to present modular arithmetic, either as "k mod n" or as $n\mathbf{Z}$ (or equivalence classes $k + n\mathbf{Z}$).

For the remainder, we'll focus on subobjects.

2. Converting between presentations

"Solving" an equation (or system of equations) is exactly going from an implicit description of an object to an explicit description.

For example:

- enumerating roots of a polynomial
- parametrizing a curve
- giving a basis for the kernel of a linear map.

The dual process, going from an explicit presentation to an implicit one, is less familiar: it's giving *defining* equations for an object.

One can also convert between different explicit descriptions (or implicit descriptions): one may show that two different explicit descriptions are equal by a direct computation, or showing that they satisfy the same (defining) equations, or are both aspects of some other presentation, etc.

2.1. Not better or worse: different. The importance (both theoretical and practical) of solving equations equations may lead one to think that implicit descriptions are *bad* and explicit descriptions are *good*. Instead, one should think of them as different perspectives on the same underlying phenomenon, each suited for different purposes.

For instance, if you were asked "is $(3/5, 4/5)$ in the image of $(\cos t, \sin t)$", you'd need to work with inverse trig functions to solve it, but given the implicit description $x^2 + y^2 - 1 = 0$, you could just plug it in: implicit descriptions are very good for testing points.

On the other hand, it's very easy to graph a parametric curve (by simply plotting points), while plotting a curve given implicitly is much trickier.

3. Computations in different presentations

Some further examples, these of computations in different presentations.

Given a submanifold $M \subset \mathbf{R}^n$:

Computing coordinates on the tangent or normal space:

Explicit	tangent (push forward usual tangent vector field)
Implicit	normal (grad; pull back tangent vector field on target)

To compute the other (for instance, the tangent space to an implicitly defined submanifold), take the perp.

Optimization:

Explicit	pull back to domain
Implicit	Lagrange multipliers

In linear algebra, solving a system of linear equations is going from implicit to explicit; because of duality, the dual problem "find equations that these vectors satisfy" is the same problem on the dual space:
"Equations these vectors satisfy" = "covectors that satify the dual equation"

4. Axioms and theories

More abstractly, *axioms* are an *implicit* description of a phenomenon, and classification theorems are *solutions* for systems of axioms.

Similarly, different axiomatizations are different "defining properties" for a particular object of study, and trying to come up with a good definition / good axiomatization of an object is solving the "explicit to implicit" problem.

The classic example is Euclidean geometry and the parallel postulate (Euclid's fifth postulate), which was suspected to be redundant for millenia: the phenomenon was Euclidean geometry, and the axiomatization was Euclid's axioms.

There is a duality between axioms and phenomena, as evidenced by pathological examples: given a phenomenon, you can try to axiomatize it, and then, given a system of axioms, study objects in the resulting theory, and see if they are exactly the phenomenon you started with (or are well-behaved). The axiomatization of a continuous function led to pathological examples, like Weierstrass's nowhere differentiable function. Similarly non-Euclidean geometries.

As an illustrative example (which is non-intuitive enough to remind us that axiomatizations are non-trivial), consider the axioms for a Jordan algebra. It should be something that "behaves like"

$x.y := \frac{1}{2}(xy + yx)$, but why should that be axiomatized by $x.y = y.x$ and $(x.y).(x.x) = x.(y.(x.x))$ (the Jordan law looks weird[2])?

As other examples, consider the axioms for a triangulated category, or more presently, the work on axiomatizing n-categories.

5. Abstraction and constructibility

One reason that abstraction has been important in math (especially starting in the 19th century) is because many things are *not* constructible.

For instance, solutions to a general quintic cannot be constructed from rationals using roots[3], and similarly solution to differential equations generally are not combinations of elementary functions.

More subtly, a general real number is not computable, nor is a general continuous function.

Thus the failure of explicit descriptions requires one to work abstractly, with implicit descriptions: rather than writing down the solution and studying its properties, what can you say about properties that a solution must satisfy?

In axiomatic language, what can you deduce from axioms?

An intermediate case happens in Galois theory: you can still write down a number field explicitly (you don't need to work with an abstract field), but you cannot write down elements explicitly (in terms of rationals). Rather, you can only write down elements explicitly in terms of a root, which is an *implicitly* defined algebraic number.

[2]Indeed, there are other axioms that $(xy + yx)$ satisfies that Jordan algebras don't, by work of Glennie. This is similar to Tarski's "High school algebra problem"; see John Baez's weeks 172 and 192 for details.

[3]Indeed, even a root $\sqrt[n]{x}$ is defined implicitly, as a solution to $x^n = a$.