How to Write Math

Good math writing has a certain style. It varies a little between people, but is broadly consistent. It's similar to good essay writing – clear, to the point, sufficiently supported – but takes some getting used to.

It's not arbitrary – it's developed over centuries and is really the best way of writing math. Further, good math *writing* encourages good math *thinking*.

The key point is to *structure* your argument (logically and visually) and connect the parts so that it's easy to follow – not just a long block of prose, or disconnected equations.

There's also certain stereotyped genres, like "the counterexample", and you need to write these in *exactly* the standard format.

It's okay to copy the format of my solutions – in fact, it's encouraged! If all your solutions read like my "model solutions", you're half-way done (getting it *right* is another story – but at least get proper *style*): the creativity is in the *content*, not the *form*.

Writing up

Scratch work

I do not want to see your scratch work, and many of the problems will require it.

You should *first* work out a problem on scratch paper until you have solved it, and *then* "write it up" on new, clean paper.

This results in much better homework, better understanding, and better grades – when was the last time you handed in the first draft of an essay, unproofread?

If you try to combine these into one step (as we likely all did in high school), your work will be worse, your homework will be covered in erasures or crossed out work, and you won't learn as much. You also won't save time – it's faster to do scratch work when you're not worried about messing up your final draft, and it's faster to write up a final draft when you've already hashed out the details.

Show your work

I don't want to see your scratch work, but I do want you to *prove* and *explain* your results.

If I say "compute this integral", it's not enough to just write the answer – you have to "show your steps".

However, you should not "show your screw-ups" – just the final, clean path to the answer.

If you can only answer a question partially, make what you *do have* as clean as possible: accentuate the positive.

Model answers

Your solutions to problems should "read" well. It should not be a mess of symbols and figures, left to the reader to figure out, but it also shouldn't be a paragraph of text: it should be a reasoned argument, using words, equations and pictures as appropriate. It'll take a while to learn the proper style.

Here's an (easy) example:

Question:

A 2 meter long ladder rests on flat ground and is leaning against a vertical wall, forming an angle of 60° with the ground. How high up the wall does the ladder reach?

Unacceptable answer 1:

$\sin 60 = x/2 \; x = 2\sin 60 = \sqrt{3}$

[**Commentary:** What's going on? The equations are the correct ones, but they lack explanation and are crammed together illegibly. Also, this sloppy student dropped the ° from 60°.]

Unacceptable answer 2:

The configuration of ladder, wall, and ground manifestly form a so-called "Pythagorean" or "right" triangle. Such figures are amenable to the tools of trigonometry, most presently the sine function, denoted sin. Recalling that in a "right" triangle, the sides satisfy the relation expressed by the ratio: sine of an angle equals the quotient of the length of the side opposite that angle by the hypothenuse, and using the well-known fact that sine of sixty degrees is half the square root of three, we see that

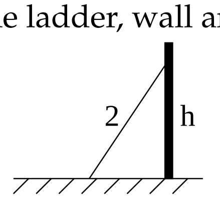
$$\frac{\sqrt{3}}{2} = \frac{h}{2}$$

where h denotes the height (in meters, natch) at which the ladder intercepts the wall. Cross-multiplying and solving, we see that the height is the square root of three meters.

[**Commentary:** Some people actually do this, and the ancients wrote like this. Use symbols – more words don't always add clarity. Also, we assume that words are used in a technical sense – there's no sense in quotes.]

Good answer 1:

The ladder, wall and ground form a right triangle:



By definition of sine, the height h satisfies

$$\sin 60^\circ = \frac{\mathrm{h}}{2}$$

Solving, we get:

$\boxed{\mathrm{h} = 2\sin 60^\circ = \sqrt{3}\text{ meters}}$

[**Commentary:** This is clear and to the point: a good balance of words and equations. Note the use of good notation (h for height), and a picture to clarify. Also, in questions with a final answer, it's a good idea to box your answer. Lastly, in a physics class one should probably take more care about units, but there's no need in a math class.]

Good answer 2:
This forms a triangle with $\sin 60^\circ = \mathrm{h}/2$, so $\mathrm{h} = \sqrt{3}$ meters.

[**Commentary:** This is okay if the reader already knows that the student can solve these problems and understands what's going on (and is sick of solving 20 of these problems), but in general you should stick with the more florid "Good answer 1": I won't be giving you many menial problems, so each problem is worth writing something for. Notice that this is not much longer than "Bad answer 1", but much clearer.]

Solutions should read more or less like prose – try reading your solutions aloud!

Style
The following are good constructions to use in writing math:

- "if ... then ..."
- connectives, like: "so", "thus", "therefore", etc.
- "such that (s.t.)"

A particularly useful construction is: "pick a such that $a > 17$", which tells me that a is some constant, chosen arbitrarily and satisfying $a > 17$.

Connectives
Words and phrases like **so**, **thus**, **therefore**, **this implies**, **we get**, **we obtain**, **we see**, **it follows that**, **because** (and so forth) really help a solution flow better.

Please avoid the symbols $\therefore$ $\because$ and $\ni$ (which some use, meaning: "therefore", "because", and "such that"), as they're hard to read (especially the dots). If you must be brief, write "so", "b/c", and "s.t." – these are far more legible.

The symbol $\implies$ (for "implies") is okay, but so is "so".

[Yes, there's a certain geeky chic in writing solutions using only symbols, but please use words.]

Proof and disproof
Proofs and disproofs are conceived and written up rather differently.

Consider the definition of an even function:
"f is even if and only if $\forall x$, $f(x) = f(-x)$"

Thus, to show that f is even, you need to show that for *all possible* x, $f(x) = f(-x)$.

To show that f is *not* even, you only need to produce *one specific concrete counterexample*.

Example of a proof
$f(x) = 3x^2 + 7x^4$ is even, as

$$f(-x) = 3(-x)^2 + 7(-x)^4 = 3x^2 + 7x^4 = f(x)$$

[**Commentary:** Note that we didn't use any specific x: we manipulated the formulae.]

Example of a disproof
$f(x) = 3x + 7x^4$ is not even, as

$$f(1) = 3(1) + 7(1)^4 \qquad = 10$$

but

$$f(-1) = 3(-1) + 7(-1)^4 \qquad = 4$$

(and $4 \neq 10$).

[**Commentary:** Note that we used a single numerical counterexample.]

Hint
Try *very simple* counterexamples. I could have shown that $f(7) \neq f(-7)$, but that would have been more work. Try $0, 1, 2$ before you try 17.

Discovery versus write-up
One hurdle in learning math is:
"A clean write-up, while easy to read, is *not* how you actually *solve* the problem – but it is how you *present* your solution."

To underline: solve the problem in your scratchwork, but present a clean solution, not your intermediate steps.

Here's an example of discovery and scratchwork, using the previous problem:
Scratchwork
"Hmmm…is $f(x) = 3x + 7x^4$ even?

Well, let's try $f(-x) = 3(-x) + 7(-x)^4 = -3x + 7x^4$.
That sure looks different to me – I wonder if I can find a specific counterexample.
Let's try $f(1)$ and $f(-1)$. Well, $f(1) = 3(1) + 7(1)^4 = 10$, and $f(-1) = 3(-1) + 7(-1)^4 = 4$, and these aren't the same – success!"

That's good, and it's what you should be *thinking*, and it's what should show up on your *scratch work*, but *do not* write it on what you hand in.

Instead, write what I showed above:

Write-up

$f(x) = 3x + 7x^4$ is not even, as

$$f(1) = 3(1) + 7(1)^4 \qquad = 10$$

but

$$f(-1) = 3(-1) + 7(-1)^4 \qquad = 4$$

(and $4 \neq 10$).

Detail

There is a *correct* level of detail in a solution to a problem; there is such a thing as saying too little *and* saying too much. This is not a rigid standard ("This proof takes exactly 17 words"), but there is an acceptable range: if you are an outlier, you are *not* a genius (for being so brief) or a thorough auteur: you're simply being terse or verbose.

In exposition, the correct detail says everything that is necessary (and reminds the audience of points they may have forgotten), but does not obscure the main point with routine, tedious detail. This is easiest if you *know your audience*. For problem sets, that means determining how much detail your grader wishes to see, and providing that.

Saying too little versus saying too much is asymmetric: if you leave out *necessary* steps, the proof has a hole, and it may be wrong (or the reader may not be able to fill in the gaps); while if you write too much, the reader can ignore the excess (or simply be bored to tears) – thus it is reasonable to err on the side of saying too much[1].

However, students should be corrected if they say too much or too little. Part of understanding a problem is understanding the relative importance of steps: what are the key points, and what is routine?

For instance, you should not justify algebraic operations ("by commutativity; by associativity") when solving routine equations, but you *should* justify them when deducing properties of a new algebraic structure (or deducing rules of arithmetic from first principles).

[1] In other situations, such as when someone tunes out after the second sentence, saying too much is much worse, but in rigorous proofs, it's better to say too much.

As a rule of thumb, the first time you give an argument, you should do it, and do it in gory detail. Successive times (after you've shown that you know how to do it, and it really has become routine), you should omit the details. This isn't optional: once it's routine, the reader really doesn't want to see the details if they're exactly like the previous time.

Concretely, a grader *should* mark when there is too great or too little detail, and if you continue, should deduct points: nudge gently at first, then more firmly.

This isn't as big a point as learning material, but it is an important meta-point: it's not *just* exposition and style, but it's also learning to think about material and understand it more deeply.

I mention this because I have had problem sets that have varied by an order of magnitude in length: one student handed in a 1-page set, another handed in a 12-page set. The first was too terse, the second was too long.

Techniques

WLOG

A somewhat abused technique is WLOG:
"assume, without loss of generality, (WLOG)"

The point is that you can only do this when it really isn't a loss of generality, which you can usually do when there is some symmetry in the problem.

A good WLOG: when wanting to prove something about $\sin\theta$, you take $\theta \in [0, 2\pi]$: this is WLOG b/c of the periodicity of sin.

Actual example: when proving Fermat's theorem on stationary points, which can be stated as: "If $f'(x_0) \neq 0$ and $x_0 \in (a, b)$, then x_0 is not a local extremum", you can and should say "WLOG, take $f'(x_0) > 0$" – as the discussion is identical for $f'(x_0) < 0$, just with signs reversed.

Not okay WLOG: given $x^2 > 0$, so WLOG take $x = 7$. Here $x = 7$ is a *special case*: it is a loss of generality!