CHAPTER 1

How to learn math, and what that means

Learning math is not hard, but requires time, and hence commitment: it's on the same order as learning to play the piano.

In a nutshell, you need a teacher and materials: contact a local university math department, and say:
"I want to learn Galois theory; could you suggest a class or tutor?"

For materials, I suggest Dummit & Foote's "Abstract Algebra" (a shorter, more direct book is Artin's "Algebra"). You might also look at MIT OpenCourseWare courses 18.701 and 18.702.

You may need to do some preparatory work; I would suggest basic algebraic number theory.

You can learn math on your own, but feedback is extremely useful (especially at first), and texts are generally not optimized for self-study. As with anything, until you develop internal standards and feedback loops (called alternatively "taste", "style", or "maturity"), learning is very rough going.

Math takes time and effort to learn, but I find the learning fun and the rewards exquisite, and I encourage everyone to take the time to learn it.

1. Why should I learn math?

Math is beautiful, useful (it both builds the brain and teaches particular skills and knowledge), and as valuable a part of our common culture as music or poetry.

I think there no better subject to study.

It is unfortunately a large hole in education, particularly US education: math is taught, but poorly and at a low level. This is not unique to math – philosophy is also little taught, and music likewise (mostly informally) – but my love of math makes this failure of education hurt all the keener.

I cannot here convey the joy nor beauty of math: it is an unfamiliar beauty, yet deeply human and underlies many other more familiar beauties. Give it a shot, and you may find a new love that'll last a lifetime.

By unfamiliar I mean: explaining the joy and beauty of math to someone for whom "math" means "long division" is like explaining poetry to someone for whom "English" means "spelling". If you have never read a sentence, the beauty of poetry is literally indescribable: it is completely alien. This is truly how different serious math is from arithmetic.

Math is not primarily about doing computations, but about abstract *patterns* and *structures*. Computations are a necessary part of doing math, like spelling is to writing, but are not an end into themselves.

Some of these structures have the character of wit, of cleverness; others are geometric, but a geometry that cannot be drawn and can only be visualized indirectly, in your mind's eye.

2. What do you mean, "learn math"?

I primarily mean "learn to think, read and write abstractly and rigorously".

I identify 2 dimensions of "learning math": depth of thought, and breadth of material.

Within "depth of thought", I experienced 2 major stages of learning: basic rigor, and high abstraction. (These terms are idiosyncratic.)

Basic rigor is the main step in learning math, and is the biggest hurdle: it takes 500–1,000 hours in usual teaching (regular effort for 1 or 2 years: 1 or 2 full year math classes, or equivalent), probably less with tutoring. It is demanding for the student and demanding for the teacher – after this, everything is cream: not effortless, but achievable. The result is "mathematical literacy": being able to grasp and work with a definition, have a mathematical conversation, read a proof, write a proof, solve a problem. Once one has achieved this, most of math is open to you, and there is a great deal there.

By "high abstraction" I mean specifically higher algebra, especially category theory. Notions like "naturality" and "adjoint functors" require considering not just a single object, but a whole collection of objects at once, and require more demanding abstract reasoning. Similarly abstract (in a similar way) is proof theory.

This is not necessary for good and deep math: it was absent before the 20th century, and one can do very good math today without this level of abstraction. However, I find this depth of thought valuable and enjoyable, and it is essential in some areas of math (notably modern algebraic geometry).

Regarding breadth: math is a vast area (this is not at all apparent from most schooling). There is some unity across fields, but different fields are very different in feel: they study different object and use different approaches. Mathematicians generally enjoy some areas (theirs and related) and enjoy others far less, at times even criticizing other areas of math as worthless. In other essays, I'll indicate particularly central or beautiful areas.

It is completely unreasonable to expect to gain a good grasp of even the basics of all major areas of mathematics: while enjoyable, this would easily take decades, because it takes long work to grasp math: you cannot speed through it.

Lastly, some areas of math are very deep, requiring years of study of several areas before you can even understand what is being studied; the Langlands program is a great example. To understand such a field, you must learn basic fields, then other fields that build on these, and so forth for several levels.

3. Can I learn math?

Learning to do math absolutely does not require special talent. Many people (particularly in the US) have had awful experiences with math, which goes by such names as "math anxiety" or "math phobia". If you suffer from these, you'll need a sensitive teacher (effectively, some form of therapy), and you may prefer to direct your energy elsewhere than math.

Difficulties learning math tend to come either from a particular teaching style being incompatible with your favored learning styles, or just shoddy teaching: it's not a judgement on you or on math.

Having special talent or burning interest makes learning anything easier, but is not *necessary* unless you wish to compete at the highest levels. You can have a wonderful time (and indeed be a quite productive mathematician) without special gifts – just general intelligence and dedication.

4. Goals

You may find it helpful to have a goal when studying math, else you'll get lost. I've suggested Galois theory.

Here are some others you may like (several of which have a mystique); I do not know good references for most of these.

Gödel's incompleteness theorems: Smullyan argues that if you are interested in philosophical consequences, you actually want to study Tarski's Indefinability Theorem, which is both simpler and more philosophically meaningful.

The definitive (and light) reference is "Gödel's Proof" by Nagel and Newman.

A less technical (and apparently entertaining) book on this theme is "Gödel, Escher, Bach: An Eternal Golden Braid" by Douglas Hofstadter

A more technical account is "Gödel's Incompleteness Theorems", by Raymond Smullyan, which is apparently the definitive guide to understanding the result and context.

General relativity: Many find physics a motivation for learning math, and many enjoy the physical interpretation ("This isn't just abstract math – it describes the structure of the universe, and black holes!"). If this sounds like you, consider learning about general relativity.

General relativity involves curved space-time (in math terms: pseudo-Riemannian manifolds, a part of differential geometry), which is a pretty area of math, and it's apparently a manageable goal.

Quantum physics: This involves fancier and more interesting math than relativity, as far as I know. Understanding the math underlying the standard model seems a good goal, though perhaps ambitious.

The Prime Number Theorem: Roughly, this says that the probability that a number n is prime is $1/\log n$.

Amazingly, this statement about primes was first proven (and is best understood) by looking at a function on the complex numbers, ζ – the famous Riemann ζ-function, which also features in the Riemann Hypothesis: the Riemann Hypothesis implies a much stronger result than the prime number theorem. This is the fundamental result in "analytic number theory" (using calculus to prove results about primes).

5. Resources

The best resources are people: ask a mathematician what would be interesting or useful to study and how, and you'll often get a thoughtful answer. Wikipedia and its myriad articles can point you in myriad directions, though for a specific topic, textbooks can give far more detail and exercises. For finding textbooks, Amazon reviews are often useful, as are mathematicians.

6. What is Galois theory?

First, Galois theory proves that there is no quintic formula analogous to the quadratic formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$: there are degree 5 polynomials whose roots (zeros) cannot be expressed in terms of radicals (square roots, cube roots, etc.).

More generally, Galois theory is the study of fields and their symmetries: a field is a set of numbers that you can add, subtract, multiply and divide (familiar examples are the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, and the complex numbers $\mathbf{C}$).

The basic idea of Galois theory is that fields have symmetries, and similarly, that roots of polynomials have symmetries. A familiar example is complex conjugation, which interchanges the two roots i and $-i$ of the polynomial x^2+1. *Algebraically*, you can't distinguish i and $-i$: they're both just some number that squares to -1. More familiarly, $\sqrt{2}$ and $-\sqrt{2}$ are both algebraically simply a number that squares to 2.

Thus if the roots of a polynomial can be expressed by radicals, the symmetries of those roots (we call this the "group of symmetries") are built up from symmetries of radicals, which are very simple.

There are degree 5 polynomials whose roots have the same symmetries as the icosahedron (!), which is a more complicated group and cannot be built up from symmetries of radicals. That's the key idea: making this precise is both necessary and interesting.

For me, the most exciting point is not solving polynomials (though proving that no formula exists is impressive), but that there is a *hidden structure* which is dictating the behavior – and that the structure is in some sense *geometric* (roots of polynomials are like an icosahedron??).

The field is named after Évariste Galois, a tragic French mathematician, and was developed in the first half of the 19th century.

7. Why Galois theory?

beautiful & central: Calculus and linear algebra are central, but not so beautiful; many things are beautiful but less central.

accessible: Galois theory is discrete math (no limits, as in calculus), computational, and familiar: it involves working with polynomials.

natural starting point and natural ending point: Galois theory is a starting point for further study in mathematics: abstract algebra is itself very interesting, and relates to many other areas, notably algebraic geometry and algebraic topology.

Conversely, it is a natural ending point: it explains the statements you may have heard whispered that "you can't trisect an angle" or "you can't solve the quintic", and gives closure to Euclidean geometry and basic statements about polynomials. Beware that if you go further, there are few natural stopping points: a vast sea is open, and even the basic questions are either unanswered or have answers that take a lifetime to understand (like the classification of finite simple groups).

Historically, the 19th century was a watershed in mathematics: many old questions were solved, and vast new areas opened up as a result. Galois theory was a key part of this.

Introduction to cutting edge math: From Galois theory, it's not too large a stretch to study automorphic forms, which connect to the Langlands program and Wiles's proof of Fermat's Last Theorem: serious, cutting edge math.

Best wishes, and enjoy!