

## CHAPTER 1

### Trace

The trace is a natural map  $\text{End } V \rightarrow K$ , coming from the pairing  $V^* \times V \rightarrow K$ .

This is not a map of algebras, but it is a map of Lie algebras (in particular linear); it is a sort of counit for  $\text{End } V$ .

In coordinates, it's "sum of diagonal entries",

$$\text{tr } A = \sum_i a_{ii}.$$

In Einstein notation,  $a_i^i$ .

It is also the sum of the eigenvalues, and can be interpreted as "number of fixed points, with weights".

Geometrically, it is the derivative of the determinant (at the identity), and is infinitesimal change in volume.

#### 1. Abstract definition

There is a natural pairing  $V^* \times V \rightarrow K$ , namely "evaluate a covector on a vector". This is bilinear, so it yields a map on the tensor product,  $V^* \otimes V \rightarrow K$ . Now compose with the natural isomorphism  $\text{Hom}(V, V) = V^* \otimes V$ , which yields

$$\text{End } V = V^* \otimes V \rightarrow K.$$

This map is the trace.

#### 2. Relation to multiplication, using Einstein notation

The pairing  $V^* \times V \rightarrow K$ , and thinking of operators as elements of the tensor product (via  $\text{End } V = V^* \otimes V$ ), occur in a number of basic operations, more familiar than trace. We illustrate, using Einstein notation.

<b>interpretation</b>	<b>matrix interpretation</b>	<b>Einstein notation</b>
inner product	pair covector with vector	$a_i b^i$
apply a map to a vector	multiply vector by matrix	$a_j^i c^j$
compose maps	multiply matrices	$a_j^i b_k^j$
trace	trace	$a_i^i$

**in terms of maps**

$$V^* \times V \rightarrow K$$

$$V \times \text{Hom}(V, W) \rightarrow W$$

$$\text{Hom}(U, V) \times \text{Hom}(V, W) \rightarrow \text{Hom}(U, W)$$

$$\text{End } V \rightarrow K$$

**in terms of tensor product**

$$V^* \otimes V \rightarrow K$$

$$V \otimes V^* \otimes W \rightarrow W$$

$$U^* \otimes V \otimes V^* \otimes W \rightarrow U^* \otimes W$$

$$V^* \otimes V \rightarrow K$$

As a more sophisticated example, given any mixed variance tensor, one gets lower order tensors by contraction: pair a covector against a vector (a  $(k, l)$ -tensor contracts to yield a  $(k-1, l-1)$ -tensor; one must choose which indices to contract). In Einstein notation, pair an upper and a lower index. A key example is the Ricci curvature, by contraction of the Riemannian curvature.

Note that a tensor is usually interpreted as a form

$$V^* \times \cdots \times V^* \times V \times \cdots \times V \rightarrow K$$

so if one contracts on the left, one must argue why the form descends.

(fixme: a commutative diagram would help)

More naturally, dualize by taking a vector and covector to the right side, and then contract there (or rather, interpret the tensor as taking in  $(k-1, l-1)$  (covectors, vectors), and outputting  $(1, 1)$  (covector, vector), aka an operator):

$$V^* \times \cdots \times V^* \times V \times \cdots \times V \rightarrow V \otimes V^* \rightarrow K$$

(the left map is the original tensor, partly dualized, the right map is pairing). Thus contraction of a tensor is somewhat indirect, depending on how you interpret the tensor.

Einstein notation also explains why “sum of diagonal entries” is a natural thing, while “sum of anti-diagonal entries” or “sum of 1st column” isn’t: trace is a natural pairing, while these others are not.

### 3. Interpretation in terms of fixed points and sum of eigenvalues

The trace is transparent in terms of a function on  $V^* \otimes V$ , but a priori opaque in terms of a function on  $\text{End } V$ .

It can be interpreted as “number of fixed points (with weights)”: given a map on a set  $X \rightarrow X$ , you get an operator on the free vector space on that set, and the trace is the number of fixed points: the  $i$ th diagonal entry is 1 iff  $x_i \mapsto x_i$ , and zero otherwise. This is most familiar for permutation matrices. Note that this is intrinsic, and doesn’t depend on an order on the set.

For a general map (not coming from a map on a set), the trace is the sum of eigenvalues (proof: invariant under change of basis; put in Jordan form (or just Schur form: upper triangular), so diagonal

entries are exactly the eigenvalues); the eigenvalues can be thought of as a sort of “weight” of fixed points.

## 4. Properties

**4.1. Lie algebra map.** Trace is a map of Lie algebras, but not a map of algebras.

Concretely,  $\text{tr } AB = \text{tr } BA$ , but in general  $\text{tr } AB \neq \text{tr } A \text{tr } B$ .

Categorically, this is because trace and composition are both the pairing of vectors and covectors: given two operators  $A, B$ , you get an element of

$$\text{End } V \otimes \text{End } V = V^* \otimes V \otimes V^* \otimes V.$$

The multiplication  $AB$  pairs the inner pair, then trace pairs the outer pair, while multiplication  $BA$  pairs the outer pair, then trace pairs the inner pair, thus these are equal.

In Einstein notation<sup>1</sup>, they’re both  $a_j^i b_i^j = b_i^j a_j^i$ .

Conversely,  $\text{tr } A \text{tr } B = \text{tr } B \text{tr } A$  pairs the left pair and the right pair, and in Einstein notation is  $a_i^i b_j^j$ .

(fixme:  $V^* \times V$  etc. with over/under braces to indicate the pairing would help)

As a concrete counter-example showing  $\text{tr } AB \neq \text{tr } A \text{tr } B$ , let  $A = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then  $\text{tr } A = 0$  but  $\text{tr } A^2 = 2!$

Another example is:  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ .

For 3-fold products, it is invariant under cyclic permutation, which is called the *cyclic property* of the trace, but not under arbitrary permutation; this can be seen by Einstein notation or drawing  $V^* \times V \times V^* \times V \times V^* \times V$  as a hexagon; then  $\text{tr } ABC = \text{tr } BCA = \text{tr } CAB$  are all contracting the same 3 sides, while other permutations pair different pairs.

(fixme: pretty hexagonal diagram, bitte?)

We can interpret  $\text{tr } AB = \text{tr } BA$  as  $\text{tr}[A, B] = 0$ , so it vanishes on the derived algebra. Since  $K$  has a commutative Lie algebra structure (trivially), the trace is a map of Lie algebras.

**4.2. Transpose.** Given a matrix, its trace equals the trace of the transpose:  $\text{tr } A^T = \text{tr } A$ .

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<sup>1</sup>You obviously can switch  $i$  and  $j$ ; they are dummy indices.

This is obvious from the matrix characterization (as the diagonal is invariant under transpose); more intrinsically, the trace of the adjoint<sup>2</sup> of a map equals the original trace: trace commutes with the duality isomorphism  $\text{End } V = \text{End } V^*$ .

Written in terms of tensor products,

$$\text{End } V = V^* \otimes V = V \otimes V^* = V^{**} \otimes V^* = \text{End } V^*$$

and the trace is the same pairing: you've just reversed the order of the terms.

(fixme: a commutative diagram would help)

### 5. Geometric interpretation

Geometrically, the trace is the infinitesimal change in volume:  $\text{tr} = \det'_1$ . Trace is the derivative of the determinant at the identity.

The fact that it's not multiplicative, but is a map of Lie algebras suggests looking at the Lie algebra, whose geometric interpretation is infinitesimals. The determinant is a map of Lie groups, and the trace is a map of Lie algebras, so it's not surprising that they should be related in this way.

The kernel of the trace (the trace-free operators) is  $\mathfrak{sl}$ , the basic simple Lie algebra; this is the Lie algebra of the kernel of the determinant,  $\text{SL}$ .

I discuss their connection more in "Trace and Determinant: a Lie theory perspective".

### 6. Hopf algebra?

The endomorphism ring of a finite dimensional vector space seems a lot like a Hopf algebra, especially if there's an inner product (dual/transpose being an antipode), but I can't prove this and can't find any references on it.

From this point of view, trace is the counit  $A \rightarrow K$  and scalar matrices is the unit  $K \rightarrow A$ .

### 7. Etymology

The term comes from German, where it is called *Spur* (cognate to English "spoor"), notated *Sp*: the trail of an animal (via its droppings).

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<sup>2</sup>For complex operators, we just mean the algebraic dual, the transpose: no conjugation.

## 8. Applications

A key application is group characters: given a group representation, its trace is called the character, and the characters are very useful for understanding the group.

One also defines a trace for operators on Hilbert spaces and Banach spaces, but it can't be globally defined, hence one has *trace class* operators: the ones for which trace can be defined.