CHAPTER 1

Projective Structure on index p normal subgroups

Index p normal subgroups[1] of a group are a projective space over $\mathbf{Z}/p$, namely

$$\mathrm{Hom}(G, \mathbf{Z}/p) \setminus \{0\} / (\mathbf{Z}/p)^*.$$

This is particularly interesting to me because finite group theory often feels very un-geometric and very formal, but apparently has deep geometric structure, as this example illustrates.

1. Corollaries

Given 2 index p normal subgroups, you get a projective line of $p+1$ subgroups, and we have notions of colinear, coplanar etc.

The number of index p normal subgroups is the size of the projective space, $1 + p + \cdots + p^n = \frac{p^{n+1}-1}{p-1}$.

In particular, the number of index 2 normal subgroups must be $0, 1, 3, 7, \ldots$: it cannot be 2, for instance.

2. Proof

Maps $G \to \mathbf{Z}/p$ form a vector space (Hom inherits a structure from its target).

Given a non-zero map $f\colon G \to \mathbf{Z}/p$, its kernel is an index p normal subgroup. Its kernel is unchanged under multiplying f by a unit in $\mathbf{Z}^*$, so we get a map from $\mathbf{P}(\mathrm{Hom}(G, \mathbf{Z}/p))$ to index p normal subgroups.

Conversely, given an index p normal subgroup $N \triangleleft G$, the quotient is a group of order p, which is isomorphic to $\mathbf{Z}/p$, so we get a (non-zero) map $G \to G/N \cong \mathbf{Z}/p$. The isomorphism is not unique; it is only defined up to $\mathrm{Aut}(\mathbf{Z}/p) = \mathbf{Z}^*$ (isomorphisms are a torsor over automorphisms); thus we get the inverse map from index p normal subgroups to non-zero maps $G \to \mathbf{Z}/p$, up to $\mathbf{Z}^*$.

These maps are obviously inverse: a class of maps gives a kernel, a kernel gives a class of maps.

[1]This is the conventional order of adjectives.

3. Concretely for index 2

Given two distinct index 2 subgroups[2], their symmetric difference is another index 2 subgroup; it's the third subgroup in the line they define.

Each subgroup defines a map $G \to \mathbf{Z}/2$, and their sum is a new map $G \to \mathbf{Z}/2$. Over $\mathbf{Z}/2$, sum of characteristic functions of sets is the characteristic function of the symmetric difference (or in this case the function is 1 minus the characteristic function of the subgroup).

For $p \neq 2$ you take some combinations of p intersections of the cosets of the subgroups to get the other points on the projective line they define.

Indeed, for index 2 we can almost think of this as a group structure; geometrically there are 3 points on a line, so given any 2 distinct points, you get the third. (Hom is a group, and we're just removing the identity; $(\mathbf{Z}/2)^* = 1$ so we're not modding out). It doesn't quite work because you can't add a point to itself (you get zero, which we've taken out).

[2] Which are necessarily normal.