

CHAPTER 1

Point-wise models of the lower floors of the Postnikov Tower, with Torus Bundles

Just as covering spaces kill π_1 , one can kill factors of \mathbf{Z} in π_2 via torus bundles. This is a geometric form of the first stages of a Postnikov tower, with $S^1 = K(\mathbf{Z}, 1)$.

1. Details

Disclaimer: I might have the indexing and terminology wrong. Also, you need suitably nice spaces; I'm ignoring technicalities.

Recall that given a (connected) space X , there is a universal covering space $\tilde{X} \rightarrow X$, with fiber $\pi_1 X = K(\pi_1 X, 0)$ (i.e., the (discrete) fundamental group, thought of as an Eilenberg-Mac Lane space), where $\pi_1 \tilde{X} = 0$, but otherwise the homotopy groups agree (via the long exact sequence of a fibration, since the fiber is discrete).

In general we can kill homotopy groups in this way (from the bottom up), and this is called a Postnikov tower¹: you get a fibration

$$K(\pi_n(X), n-1) \rightarrow X_n \rightarrow X_{n-1}$$

where X_n is n -connected and the map $X_n \rightarrow X_{n-1}$ is an isomorphism on homotopy groups above n .

This construction is homotopical: the spaces X_n and the maps are only defined up to homotopy, and in general must be infinite-dimensional.

The first few stages can be done at the point-set level, which is our subject here.

2. Point-set fiber bundles

For completeness and fun, let's start at the bottom. The steps are:

- π_0 : path-connected component of basepoint
- π_1 : universal cover
- π_2 : torus bundle

¹Or Postnikov system.

2.1. Stage 0. Given a possibly path-disconnected space X , the 0th stage in the Postnikov tower is “path-connected component of the basepoint”. This admittedly feels like cheating, but $X_0 \rightarrow X$ is a bona fide fiber bundle: over points in X_0 , it has fiber a point $*$, while over points in other components, it has empty fiber \emptyset .

2.2. Stage 1. As a reminder: Given a path-connected space X , we can take its universal cover, which is a fiber bundle with discrete fiber $\tilde{X} \rightarrow X$. The fundamental group $\pi_1 X$ acts on \tilde{X} via deck transformations, and the fibers over each point are $\pi_1 X$ -torsors.

2.3. Stage 2. As a convenient non-standard notation, use f to indicate “free part”

Given a simply connected space X , we can kill the free part of its fundamental group by taking a torus bundle over X :

$$T^k \rightarrow X_2^f \rightarrow X$$

where $\pi_2 X_2^f$ is the torsion of $\pi_2 X$ (we’ve killed the free part), T^k is a torus of dimension equal to the rank of $\pi_2 X$, and $X_2^f \rightarrow X$ is an isomorphism on homotopy groups above 2.

Formally, X_2^f is 2-connected at 0, and the torus is the group quotient $\pi_2 X \rightarrow \pi_2 X \otimes \mathbf{R} \rightarrow T^k$; one might denote this $T(\pi_2^f X)$ (the torus of that free abelian group).

Note that torsion part is a canonical subgroup, while free part is a quotient subgroup ($T \rightarrow G \rightarrow F$; apologies for the abuse of T); this is reflected in homotopy groups as:

$$\pi_2 X_2^f \rightarrow \pi_2 X \rightarrow \pi_1 T^k$$

To construct this formally, take a fibration as in the Postnikov tower, and then argue that you can choose the fibration to be a genuine torus bundle.

2.3.1. Technical details. I don’t recall the usual construction of the Postnikov tower; here’s an outline of why the fibration can be taken to be a fiber bundle.

I think the obstruction to a spherical fibration being a sphere bundle lies in the hofiber of $O(2) \rightarrow HE(S^1)$, (the obstruction to a homotopy equivalence of the circle being linear; this is overkill (we just need $\text{Homeo}(S^1)$), as this gives us not just a circle bundle, but a plane bundle). This hofiber is trivial (the map $O(2) \rightarrow HE(S^1)$ is a homotopy equivalence), so a fortiori the obstruction vanishes. You can see this because $S^1 = K(\mathbf{Z}, 1)$ is a $K(\pi, 1)$, so the space of maps is h.e. to the maps of groups.

Concretely, you can linearize any homotopy equivalence, which you can do via a straight-line homotopy. Explicitly, fix a point $*$ $\in S^1$; then $\text{HE}(S^1) \rightarrow S^1$ via $f \mapsto f(*)$ is a fiber bundle with fiber two convex sets (corresponding to the homotopy equivalences 1 and -1), each component of the fiber naturally based by the linear map in that class (variationally, the minimum energy map in the class).

2.4. Summary of fiber bundles.

$$K(\pi_0 X, -1) \rightarrow X_0 \rightarrow X$$

$$K(\pi_1 X, 0) \rightarrow X_1 \rightarrow X$$

$$K(\pi_2^f X, 1) \rightarrow X_2^f \rightarrow X$$

Concretely,

$$* \rightarrow X_0 \rightarrow X$$

$$\pi_1 X \rightarrow \tilde{X} \rightarrow X$$

$$T(\pi_2^f X) \rightarrow X_2^f \rightarrow X$$

Note of course that these fiber bundles are twisted; if they were trivial, rather than killing π_k , they would add to π_{k-1} : a trivial fiber bundle with discrete fiber yields a disconnected space, and a trivial torus bundle adds to π_1 (I can't see a step 0 analog).

3. Higher generalizations

This is as far as you can go for finite dimension – you can't kill other elements of homotopy by passing to a fiber bundle with finite dimensional fiber.

You can't kill torsion elements of π_2 because $K(\mathbf{Z}/n, 1)$ is infinite dimensional lens space, and you can't kill free elements of π_3 because $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$ is infinite complex projective space. Similarly for the other higher $K(\pi, n)$.

Note that these spaces are necessarily infinite dimensional, since their homology is infinite dimensional.

Even if you wanted to do this anyway, I'm not sure that the argument that it's an actual fiber bundle (and not just a fibration) works.

Despite the above, I think I have heard of people doing this with \mathbf{CP}^∞ (to kill π_3), perhaps in TQFT?

4. Examples

A circle bundle is of this form iff the class of the fiber is null-homotopic in the total space; this includes some familiar circle bundles.

4. POINT-WISE MODELS OF THE LOWER FLOORS OF THE POSTNIKOV TOWER, WITH TORUS BUNDLES

The universal example is the tautological circle bundle on infinite complex projective space:

$$S^1 \rightarrow S^\infty \rightarrow \mathbf{CP}^\infty.$$

More suggestively,

$$K(\mathbf{Z}, 1) \rightarrow S^\infty \rightarrow K(\mathbf{Z}, 2).$$

This yields the tautological circle bundle on finite dimensional complex projective space $S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{CP}^n$, including (for $n = 1$) the Hopf bundle $S^1 \rightarrow S^3 \rightarrow S^2$.

The higher Hopf bundles are not of this form, in the sense that S^2 and S^4 are not Eilenberg-Mac Lane spaces, while the lower Hopf bundle $S^0 \rightarrow S^1 \rightarrow S^1$ is a cover, though not the universal cover.