CHAPTER 1

Kodaira dimension and curvature

The Kodaira dimension is an invariant of a variety in algebraic geometry that corresponds roughly to curvature.

This is vague; I list some connections below, particularly via classification of curves and surfaces, but I don't know the whole story. Some of the parts (the Enriques-Kodaira-Bombieri-Mumford classification of algebraic/complex surfaces, and Yau's proof of Calabi's conjecture) are *deep* theorems.

A particularly nice consequence is that algebraic invariants are defined over positive characteristic, hence we can extend our geometric intuition of curvature to these settings.

1. Definition

Kodaira dimension is generally unfamiliar to non-(algebraic geometers). The definition is somewhat technical; intuitively, it's "how many dimensions you can recover from volume forms", or "the number of independent volume forms (minus 1)".

Most simply, it's the projective dimension of the pluricanonical ring:

$$\kappa(V) := \dim \operatorname{Proj} R(V, K_V)$$

That is, let K_V be the canonical bundle (the line bundle corresponding to the canonical divisor; geometrically, the determinant bundle of regular n-forms: the top exterior power of the regular cotangent bundle). Then define a graded ring (called the *pluricanonical ring*) by letting the nth graded component be sections of the nth tensor power of the canonical bundle:

$$R_n(V, K_V) := H^0(V, nK_V)$$

Geometrically, the pluricanonical bundle nK_V corresponds to a map to projective space, called the nth pluricanonical map, and we can define the Kodaira dimension as the dimension of this image for n sufficiently large.

Algebraically, projective dimension of a ring is 1 less than its transcendence degree (think of $\mathbf{P}^n := \operatorname{Proj} K[x_0, \dots, x_n]$); from this point of view, Kodaira dimension is “number of independent volume forms (minus 1)”.

If the canonical divisor (and pluricanonical divisors) are not effective, so there are no regular volume forms, we conventionally take the Kodaira dimension to be $\kappa := -1$; some use $\kappa := -\infty$ (from the “dimension of Proj” point of view, it’s whatever you consider the dimension of the empty set, and the convention $\dim \varnothing = -\infty$ respects additivity of dimension under products). Splitting the difference, we’ll often simply say “negative Kodaira dimension”.

1.1. Basic Interpretation. Varieties of low Kodaira dimension are “special” (they are quite restricted); varieties of maximal Kodaira dimension are called “varieties of general type”, and are less restricted: a generic variety is of general type, as the name suggests.

This will be illustrated in the classifications below.

2. Connection to curvature

This is “as far as I can tell”.

Negative Kodaira dimension corresponds to positive curvature (in some direction, not necessarily all directions), zero Kodaira dimension corresponds to flatness, and maximum Kodaira dimension (general type) corresponds to negative curvature (I don’t have an intuition for Kodaira dimension between 0 and maximum; based on surfaces, I’d guess it corresponds to “negatively curved in some directions, flat in others”).

The specialness of varieties of low Kodaira dimension corresponds to the specialness of Riemannian manifolds of positive curvature (think of theorems on pinched sectional curvature, like the 1/4 pinched sphere theorem or Cheeger’s finiteness theorem), and general type corresponds to the genericity of non-positive curvature (think of Joachim Lohkamp’s theorem (Annals of Mathematics, 1994) that every manifold of dimension at least 3 admits a negative Ricci curvature metric).

More topologically, negative Kodaira dimension corresponds to positive Euler characteristic, zero Kodaira dimension to zero Euler characteristic, and positive Kodaira dimension to negative Euler characteristic. For positive Euler characteristic, the corresponding vanishing of (co)tangent vector fields corresponds to the non-effectiveness of the canonical divisor (there aren’t any non-vanishing fields/pluricanonical maps), while negative Euler characteristic reflects ampleness of the canonical divisor (there are lots of non-vanishing fields/pluricanonical maps).

I don't know how to show any of this directly, but it holds for the classifications of curves and surfaces, which I elaborate below.

3. (Complex) Dimension 1

For complex curves (real surfaces), the correspondence between Kodaira dimension and curvature is clear and exact:

genus	**space**	**Kodaira dimension**	**Euler char**	**curvature**
genus 0	S^2	$\kappa = -1$	$\chi = 2$	positive curvature
genus 1	T^2	$\kappa = 0$	$\chi = 0$	zero curvature
genus 2+	Σ_g	$\kappa = 1$	$\chi < 0$	negative curvature

4. (Complex) Dimension 2

The Enriques-Kodaira classification of algebraic/complex surfaces is a birational classification into families; coarsely by Kodaira dimension, then more finely. It is not as neat as the classification of curves, and is not fully understood: recall that in algebraic geometry, there are many families of varieties of a given dimension, and the situation gets rapidly more complicated as dimension increases.

I give details for (minimal models of) complex algebraic surfaces (which is what Enriques did); there are other families of non-algebraic complex surfaces (which is Kodaira's contribution) and algebraic surfaces in positive characteristic (due to Bombieri and Mumford).

I emphasize the geometric connections.

4.1. Kodaira dimension $\kappa = -1$. Positively curved.

rational surfaces	$\mathbf{CP}^2$: positively curved
ruled surfaces	$\mathbf{P}^1$ fibrations over C (birationally, $\mathbf{P}^1 \times C$): positively curved in 1 complex dimension

4.2. Kodaira dimension $\kappa = 0$. Flat.

Abelian varieties	tori, so flat
surfaces and Enriques surfaces	K3 is Calabi-Yau, so Ricci-flat; Enriques surfaces are an order 2 quotient of K3 surfaces, hence likewise
hyperelliptic (aka, bielliptic)	quotients of product of 2 elliptic curves: quotients of tori, so flat

4.3. Kodaira dimension $\kappa = 1$. Flat in one dimension, negatively curved in the other.

proper elliptic surfaces	surfaces with a fibration over a curve, with fiber almost everywhere an elliptic curve. The base curve has genus at least

2, else it's improper and covered above (ruled, tori, bielliptic). Geometrically, these are flat in the vertical/fiber direction, and negatively curved in the horizontal/base direction.

4.4. Kodaira dimension $\kappa = 2$. These are called "surfaces of general type", and are poorly understood.

All negatively curved surfaces are of this type (by process of elimination), but I don't know if these are all negatively curved.

They all have positive Chern numbers.

A product of two curves of genus at least 2 (which is negatively curved in both directions) is an example.

5. General Dimension

I don't know much about varieties in dimension above 2; here are some that I do know.

Rational varieties (projective space) have negative Kodaira dimension; geometrically, the Fubini-Study metric is positively curved (sectional curvature ranges from $\frac{1}{4}$ to 1).

Some examples with Kodaira dimension zero:

- Abelian varieties (complex tori); geometrically, these are flat.
- Calabi-Yau manifolds (like K3 surfaces); geometrically, these are Ricci-flat (that is the content of the Calabi conjecture, Yau's theorem).

6. Personal Remarks

I did my BA thesis on the Enriques classification of complex algebraic surfaces, under Joe Harris (I mostly read "Complex algebraic surfaces" by Arnaud Beauville), hence I was familiar with Kodaira dimension.

My observation of the correspondence in dimension 1 (between Kodaira dimension and uniformization), in Benson Farb's "Geometric Literacy" lecture) is what sparked this essay.