

CHAPTER 1

Boy's surface and sphere eversion

Boy's surface was discovered in 1901. Sphere eversion was proven (by Smale) in 1958, and was a surprise.

The gap of over 50 years is shocking: a form of sphere eversion is immediate from the existence of Boy's surface.

1. Sketch

Given Boy's surface, you get a generically 2-to-1 immersion of the sphere, by composing the covering map from the sphere to the projective plane with the immersion of the projective plane:

$$S^2 \rightarrow \mathbf{RP}^2 \rightarrow \mathbf{R}^3$$

Now take a normal field on the sphere (say, the natural field coming from the orientation on the sphere). Push this immersion in the direction of this field, and in the opposite direction. These give generically 1-to-1 immersions of the spheres. It's not obvious that these are regular homotopic to (round) embedded spheres¹, but they are clearly "inside out" from each other, which is the key point!

Now, actually writing down a homotopy is hard work, and visualizing what's going on is also quite difficult, but had anyone thought about the sphere immersion coming from Boy's surface, sphere eversion could have at least been conjectured in 1901, say by Boy or Hilbert.

¹Though there is an obvious candidate homotopy, namely Willmore flow.